

## DEFINITE INTEGRATION

- 1.** Let  $a$  be a positive real number such that  $\int_0^a e^{x-[x]} dx = 10e - 9$  where  $[x]$  is the greatest integer less than or equal to  $x$ . Then  $a$  is equal to :
- (1)  $10 - \log_e(1 + e)$       (2)  $10 + \log_e 2$   
 (3)  $10 + \log_e 3$       (4)  $10 + \log_e(1 + e)$
- 2.** The value of the integral  $\int_{-1}^1 \log_e(\sqrt{1-x} + \sqrt{1+x}) dx$  is equal to :
- (1)  $\frac{1}{2} \log_e 2 + \frac{\pi}{4} - \frac{3}{2}$       (2)  $2 \log_e 2 + \frac{\pi}{4} - 1$   
 (3)  $\log_e 2 + \frac{\pi}{2} - 1$       (4)  $2 \log_e 2 + \frac{\pi}{2} - \frac{1}{2}$
- 3.** Let  $g(t) = \int_{-\pi/2}^{\pi/2} \cos\left(\frac{\pi}{4}t + f(x)\right) dx$ , where  $f(x) = \log_e(x + \sqrt{x^2 + 1})$ ,  $x \in \mathbf{R}$ . Then which one of the following is correct ?
- (1)  $g(1) = g(0)$       (2)  $\sqrt{2}g(1) = g(0)$   
 (3)  $g(1) = \sqrt{2}g(0)$       (4)  $g(1) + g(0) = 0$
- 4.** If  $\int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi} - [\frac{x}{\pi}]\right)}} dx = \frac{\alpha\pi^3}{1 + 4\pi^2}$ ,  $\alpha \in \mathbf{R}$  where  $[x]$  is the greatest integer less than or equal to  $x$ , then the value of  $\alpha$  is :
- (1)  $200(1 - e^{-1})$       (2)  $100(1 - e)$   
 (3)  $50(e - 1)$       (4)  $150(e^{-1} - 1)$
- 5.** The value of the definite integral  $\int_{\pi/24}^{5\pi/24} \frac{dx}{1 + \sqrt[3]{\tan 2x}}$  is :
- (1)  $\frac{\pi}{3}$       (2)  $\frac{\pi}{6}$       (3)  $\frac{\pi}{12}$       (4)  $\frac{\pi}{18}$

- 6.** Let  $f : [0, \infty) \rightarrow [0, \infty)$  be defined as  $f(x) = \int_0^x [y] dy$  where  $[x]$  is the greatest integer less than or equal to  $x$ . Which of the following is true?
- (1)  $f$  is continuous at every point in  $[0, \infty)$  and differentiable except at the integer points.  
 (2)  $f$  is both continuous and differentiable except at the integer points in  $[0, \infty)$ .  
 (3)  $f$  is continuous everywhere except at the integer points in  $[0, \infty)$ .  
 (4)  $f$  is differentiable at every point in  $[0, \infty)$ .
- 7.** If  $f(x) = \begin{cases} \int_0^x (5 + |1-t|) dt, & x > 2 \\ 5x + 1, & x \leq 2 \end{cases}$ , then
- (1)  $f(x)$  is not continuous at  $x = 2$   
 (2)  $f(x)$  is everywhere differentiable  
 (3)  $f(x)$  is continuous but not differentiable at  $x = 2$   
 (4)  $f(x)$  is not differentiable at  $x = 1$
- 8.** The value of the integral  $\int_{-1}^1 \log(x + \sqrt{x^2 + 1}) dx$  is:
- (1) 2      (2) 0      (3) -1      (4) 1
- 9.** The value of the definite integral  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + e^{x \cos x})(\sin^4 x + \cos^4 x)}$  is equal to :
- (1)  $-\frac{\pi}{2}$       (2)  $\frac{\pi}{2\sqrt{2}}$       (3)  $-\frac{\pi}{4}$       (4)  $\frac{\pi}{\sqrt{2}}$
- 10.** Let the domain of the function  $f(x) = \log_4 \left( \log_5 \left( \log_3 (18x - x^2 - 77) \right) \right)$  be  $(a, b)$ . Then the value of the integral  $\int_a^b \frac{\sin^3 x}{(\sin^3 x + \sin^3(a + b - x))} dx$  is equal to \_\_\_\_\_.

11. Let  $f : (a,b) \rightarrow \mathbb{R}$  be twice differentiable function such that  $f(x) = \int_a^x g(t) dt$  for a differentiable function  $g(x)$ . If  $f(x) = 0$  has exactly five distinct roots in  $(a, b)$ , then  $g(x)g'(x) = 0$  has at least :

- (1) twelve roots in  $(a, b)$     (2) five roots in  $(a, b)$   
 (3) seven roots in  $(a, b)$     (4) three roots in  $(a, b)$

12. If  $\int_0^\pi (\sin^3 x) e^{-\sin^2 x} dx = \alpha - \frac{\beta}{e} \int_0^1 \sqrt{t} e^t dt$ , then  $\alpha + \beta$  is equal to \_\_\_\_\_.

13. The value of  $\int_{-\sqrt{2}}^{\sqrt{2}} \left( \left( \frac{x+1}{x-1} \right)^2 + \left( \frac{x-1}{x+1} \right)^2 - 2 \right)^{\frac{1}{2}} dx$  is :

- (1)  $\log_e 4$     (2)  $\log_e 16$   
 (3)  $2\log_e 16$     (4)  $4\log_e (3+2\sqrt{2})$

14. The value of  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2 + 4r^2}$  is :

- (1)  $\frac{1}{2} \tan^{-1}(2)$     (2)  $\frac{1}{2} \tan^{-1}(4)$   
 (3)  $\tan^{-1}(4)$     (4)  $\frac{1}{4} \tan^{-1}(4)$

15. If the value of the integral  $\int_0^5 \frac{x+[x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta$ , where  $\alpha, \beta \in \mathbb{R}$ ,  $5\alpha + 6\beta = 0$ , and  $[x]$  denotes the greatest integer less than or equal to  $x$ ; then the value of  $(\alpha + \beta)^2$  is equal to :

- (1) 100    (2) 25    (3) 16    (4) 36

16. The value of  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1+\sin^2 x}{1+\pi^{\sin x}} \right) dx$  is

- (1)  $\frac{\pi}{2}$     (2)  $\frac{5\pi}{4}$     (3)  $\frac{3\pi}{4}$     (4)  $\frac{3\pi}{2}$

17. If  $U_n = \left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2^2}{n^2}\right)^2 \dots \left(1 + \frac{n^2}{n^2}\right)^n$ , then

$\lim_{n \rightarrow \infty} (U_n)^{\frac{-4}{n^2}}$  is equal to :

- (1)  $\frac{e^2}{16}$     (2)  $\frac{4}{e}$     (3)  $\frac{16}{e^2}$     (4)  $\frac{4}{e^2}$

18.  $\int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x^2 - 44x + 484)} dx$  is equal to :

- (1) 6    (2) 8    (3) 5    (4) 10

19. The value of the integral  $\int_0^1 \frac{\sqrt{x} dx}{(1+x)(1+3x)(3+x)}$  is :

- (1)  $\frac{\pi}{8} \left(1 - \frac{\sqrt{3}}{2}\right)$     (2)  $\frac{\pi}{4} \left(1 - \frac{\sqrt{3}}{6}\right)$   
 (3)  $\frac{\pi}{8} \left(1 - \frac{\sqrt{3}}{6}\right)$     (4)  $\frac{\pi}{4} \left(1 - \frac{\sqrt{3}}{2}\right)$

20. Let  $[t]$  denote the greatest integer  $\leq t$ . Then the value of  $8 \cdot \int_{-\frac{1}{2}}^1 ([2x] + |x|) dx$  is \_\_\_\_\_.

21. If  $x \phi(x) = \int_5^x (3t^2 - 2\phi'(t)) dt$ ,  $x > -2$ , and  $\phi(0) = 4$ , then  $\phi(2)$  is \_\_\_\_\_.

22. If  $[x]$  is the greatest integer  $\leq x$ , then  $\pi^2 \int_0^2 \left( \sin \frac{\pi x}{2} \right) (x - [x])^{[x]} dx$  is equal to :

- (1)  $2(\pi - 1)$     (2)  $4(\pi - 1)$   
 (3)  $4(\pi + 1)$     (4)  $2(\pi + 1)$

23. The function  $f(x)$ , that satisfies the condition

$$f(x) = x + \int_0^{\pi/2} \sin x \cdot \cos y f(y) dy, \text{ is :}$$

- (1)  $x + \frac{2}{3}(\pi - 2)\sin x$     (2)  $x + (\pi + 2)\sin x$   
 (3)  $x + \frac{\pi}{2}\sin x$     (4)  $x + (\pi - 2)\sin x$

- 24.** The value of the integral,  $\int_1^3 [x^2 - 2x - 2] dx$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ , is :
- (1)  $-\sqrt{2} - \sqrt{3} + 1$       (2)  $-\sqrt{2} - \sqrt{3} - 1$   
 (3)  $-5$       (4)  $-4$
- 25.** Let  $f(x)$  be a differentiable function defined on  $[0, 2]$  such that  $f'(x) = f'(2-x)$  for all  $x \in (0, 2)$ ,  $f(0) = 1$  and  $f(2) = e^2$ . Then the value of  $\int_0^2 f(x) dx$  is :
- (1)  $1 - e^2$       (2)  $1 + e^2$   
 (3)  $2(1 - e^2)$       (4)  $2(1 + e^2)$
- 26.** If  $\int_{-a}^a (|x| + |x-2|) dx = 22$ , ( $a > 2$ ) and  $[x]$  denotes the greatest integer  $\leq x$ , then  $\int_a^{-a} (x + [x]) dx$  is equal to \_\_\_\_\_.
- 27.** The value of  $\int_{-1}^1 x^2 e^{[x^3]} dx$ , where  $[t]$  denotes the greatest integer  $\leq t$ , is :
- (1)  $\frac{e-1}{3e}$       (2)  $\frac{e+1}{3}$       (3)  $\frac{e+1}{3e}$       (4)  $\frac{1}{3e}$
- 28.**  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \dots + \frac{n}{(2n-1)^2} \right]$  is equal to :
- (1)  $\frac{1}{2}$       (2)  $1$       (3)  $\frac{1}{3}$       (4)  $\frac{1}{4}$
- 29.** The value of  $\int_{-2}^2 |3x^2 - 3x - 6| dx$  is \_\_\_\_\_.
- 30.** For  $x > 0$ , if  $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$ , then  $f(e) + f\left(\frac{1}{e}\right)$  is equal to :
- (1)  $1$       (2)  $-1$       (3)  $\frac{1}{2}$       (4)  $0$
- 31.** Let  $f(x) = \int_0^x e^t f(t) dt + e^x$  be a differentiable function for all  $x \in \mathbb{R}$ . Then  $f(x)$  equals :
- (1)  $2e^{(e^x-1)} - 1$       (2)  $e^{e^x} - 1$   
 (3)  $2e^{e^x} - 1$       (4)  $e^{(e^x-1)}$
- 32.** If  $I_{m,n} = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ , for  $m, n \geq 1$  and  $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \alpha I_{m,n}$ ,  $\alpha \in \mathbb{R}$ , then  $\alpha$  equals \_\_\_\_\_.
- 33.** The value of the integral  $\int_0^\pi |\sin 2x| dx$  is
- 34.** The value of  $\int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} dx$  is
- (1)  $\frac{\pi}{4}$       (2)  $4\pi$       (3)  $\frac{\pi}{2}$       (4)  $2\pi$
- 35.** If  $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x dx$ , then :
- (1)  $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}$  are in G.P.  
 (2)  $I_2 + I_4, I_3 + I_5, I_4 + I_6$  are in A.P.  
 (3)  $I_2 + I_4, (I_3 + I_5)^2, I_4 + I_6$  are in G.P.  
 (4)  $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}$  are in A.P.
- 36.** The value of  $\sum_{n=1}^{100} \int_{n-1}^n e^{x-[x]} dx$ , where  $[x]$  is the greatest integer  $\leq x$ , is
- (1)  $100(e-1)$       (2)  $100(1-e)$   
 (3)  $100e$       (4)  $100(1+e)$
- 37.** Consider the integral
- $$I = \int_0^{10} \frac{[x] e^{[x]}}{e^{x-1}} dx,$$
- where  $[x]$  denotes the greatest integer less than or equal to  $x$ . Then the value of  $I$  is equal to:
- (1)  $9(e-1)$       (2)  $45(e+1)$   
 (3)  $45(e-1)$       (4)  $9(e+1)$

38. Let  $P(x) = x^2 + bx + c$  be a quadratic polynomial with real coefficients such that

$\int_0^1 P(x)dx = 1$  and  $P(x)$  leaves remainder 5 when divided by  $(x - 2)$ . Then the value of

$9(b + c)$  is equal to:

- (1) 9      (2) 15      (3) 7      (4) 11

39. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such

that  $f(x) + f(x + 1) = 2$ , for all  $x \in \mathbb{R}$ . If

$I_1 = \int_0^8 f(x)dx$  and  $I_2 = \int_{-1}^3 f(x)dx$ , then the value

of  $I_1 + 2I_2$  is equal to \_\_\_\_\_.

40. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = e^{-x} \sin x$ . If  $F : [0, 1] \rightarrow \mathbb{R}$  is a differentiable function such

that  $F(x) = \int_0^x f(t) dt$ , then the value of

$\int_0^1 (F'(x) + f(x))e^x dx$  lies in the interval

(1)  $\left[ \frac{327}{360}, \frac{329}{360} \right]$

(2)  $\left[ \frac{330}{360}, \frac{331}{360} \right]$

(3)  $\left[ \frac{331}{360}, \frac{334}{360} \right]$

(4)  $\left[ \frac{335}{360}, \frac{336}{360} \right]$

41. If the integral  $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$ ,

where  $\alpha, \beta, \gamma$  are integers and  $[x]$  denotes the greatest integer less than or equal to  $x$ , then the value of  $\alpha + \beta + \gamma$  is equal to :

- (1) 0      (2) 20      (3) 25      (4) 10

42. Let  $I_n = \int_1^e x^{19} (\log|x|)^n dx$ , where  $n \in \mathbb{N}$ .

If  $(20)I_{10} = \alpha I_9 + \beta I_8$ , for natural numbers  $\alpha$  and  $\beta$ , then  $\alpha - \beta$  equal to \_\_\_\_\_.

43. Which of the following statements is incorrect for the function  $g(\alpha)$  for  $\alpha \in \mathbb{R}$  such that

$$g(\alpha) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

(1)  $g(\alpha)$  is a strictly increasing function

(2)  $g(\alpha)$  has an inflection point at  $\alpha = -\frac{1}{2}$

(3)  $g(\alpha)$  is a strictly decreasing function

(4)  $g(\alpha)$  is an even function

44. Let  $f(x)$  and  $g(x)$  be two functions satisfying  $f(x^2) + g(4-x) = 4x^3$  and  $g(4-x) + g(x) = 0$ ,

then the value of  $\int_{-4}^4 f(x)^2 dx$  is

45. Let  $g(x) = \int_0^x f(t) dt$ , where  $f$  is continuous

function in  $[0, 3]$  such that  $\frac{1}{3} \leq f(t) \leq 1$  for all

$t \in [0, 1]$  and  $0 \leq f(t) \leq \frac{1}{2}$  for all  $t \in (1, 3]$ . The

largest possible interval in which  $g(3)$  lies is :

(1)  $\left[ -1, -\frac{1}{2} \right]$       (2)  $\left[ -\frac{3}{2}, -1 \right]$

(3)  $\left[ \frac{1}{3}, 2 \right]$       (4)  $[1, 3]$

**SOLUTION****1. Official Ans. by NTA (2)****Sol.**  $a > 0$ Let  $n \leq a < n + 1$ ,  $n \in \mathbb{W}$ 

$$\therefore a = [a] + \{a\}$$

$$\Downarrow \quad \Downarrow$$

G.I.F Fractional part

$$\text{Here } [a] = n$$

$$\text{Now, } \int_0^a e^{x-[x]} dx = 10e - 9$$

$$\Rightarrow \int_0^n e^{\{x\}} dx + \int_n^a e^{x-[x]} dx = 10e - 9$$

$$\therefore n \int_0^1 e^x dx + \int_n^a e^{x-n} dx = 10e - 9$$

$$\Rightarrow n(e-1) + (e^{a-n} - 1) = 10e - 9$$

$$\therefore \boxed{n=0} \text{ and } \{a\} = \log_e 2$$

$$\text{So, } a = [a] + \{a\} = (10 + \log_e 2)$$

 $\Rightarrow$  Option (2) is correct.
**2. Official Ans. by NTA (2)****ALLEN Ans. (3)**

**Sol.** Let  $I = 2 \int_0^1 \underbrace{\ln(\sqrt{1-x} + \sqrt{1+x})}_\text{(I)} \underbrace{1}_\text{(II)} dx$   
(I.B.P.)

$$\therefore I = 2 \left[ \left( x \cdot \ln(\sqrt{1-x} + \sqrt{1+x}) \right)_0^1 \right]$$

$$- \int_0^1 x \cdot \left( \frac{1}{\sqrt{1-x} + \sqrt{1+x}} \right) \cdot \left( \frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}} \right) dx$$

$$= 2(\ln \sqrt{2} - 0) - \frac{2}{2} \int_0^1 \frac{x \sqrt{1-x} - \sqrt{1+x}}{(\sqrt{1-x} + \sqrt{1+x}) \sqrt{1-x^2}} dx$$

$$= (\log_e 2) - \int_0^1 \frac{x \cdot (2 - 2\sqrt{1-x^2})}{-2x\sqrt{1-x^2}} dx \quad (\text{After rationalisation})$$

$$= (\log_e 2) + \int_0^1 \left( \frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}} \right) dx$$

$$= (\log_e 2) + (\sin^{-1} x)_0^1 - 1$$

$$= \log_e 2 + \left( \frac{\pi}{2} - 0 \right) - 1$$

$$\therefore I = (\log_e 2) + \frac{\pi}{2} - 1$$

 $\Rightarrow$  Option (3) is correct.
**3. Official Ans. by NTA (2)**

$$\text{Sol. } g(t) = \int_{-\pi/2}^{\pi/2} \left( \cos \frac{\pi}{4} t + f(x) \right) dx$$

$$g(t) = \pi \cos \frac{\pi}{4} t + \int_{-\pi/2}^{\pi/2} f(x) dx$$

$$g(t) = \pi \cos \frac{\pi}{4} t$$

$$g(1) = \frac{\pi}{\sqrt{2}}, g(0) = \pi$$

## 4. Official Ans. by NTA (1)

$$\text{Sol. } I = \int_0^{100\pi} \frac{\sin^2 x}{e^{\frac{x}{\pi}}} dx = 100 \int_0^\pi \frac{\sin^2 x}{e^{\frac{x}{\pi}}} dx$$

$$100 \int_0^\pi e^{-\frac{x}{\pi}} \frac{(1 - \cos 2x)}{2} dx$$

$$= 50 \left\{ \int_0^\pi e^{-\frac{x}{\pi}} dx - \int_0^\pi e^{-\frac{x}{\pi}} \cos 2x dx \right\}$$

$$I_1 = \int_0^\pi e^{-\frac{x}{\pi}} dx = \left[ -\pi e^{-\frac{x}{\pi}} \right]_0^\pi = \pi(1 - e^{-1})$$

$$I_2 = \int_0^\pi e^{-\frac{x}{\pi}} \cos 2x dx$$

$$= -\pi e^{-\frac{x}{\pi}} \cos 2x \Big|_0^\pi - \int -\pi e^{-\frac{x}{\pi}} (-2 \sin 2x) dx$$

$$= \pi(1 - e^{-1}) - 2\pi \int_0^\pi e^{-\frac{x}{\pi}} \sin 2x dx$$

$$= \pi(1 - e^{-1}) - 2\pi \left\{ -\pi e^{-\frac{x}{\pi}} \sin 2x \Big|_0^\pi - \int -\pi e^{-\frac{x}{\pi}} 2 \cos 2x dx \right\}$$

$$= \pi(1 - e^{-1}) - 4\pi^2 I_2$$

$$\Rightarrow I_2 = \frac{\pi(1 - e^{-1})}{1 + 4\pi^2}$$

$$\therefore I = 50 \left\{ \pi(1 - e^{-1}) - \frac{\pi(1 - e^{-1})}{1 + 4\pi^2} \right\}$$

$$= \frac{200(1 - e^{-1})\pi^3}{1 + 4\pi^2}$$

## 5. Official Ans. by NTA (3)

$$\text{Sol. Let } I = \int_{\pi/24}^{5\pi/24} \frac{(\cos 2x)^{1/3}}{(\cos 2x)^{1/3} + (\sin 2x)^{1/3}} dx \quad \dots(i)$$

$$\Rightarrow I = \int_{\pi/24}^{5\pi/24} \frac{\left( \cos \left\{ 2 \left( \frac{\pi}{4} - x \right) \right\} \right)^{\frac{1}{3}}}{\left( \cos \left\{ 2 \left( \frac{\pi}{4} - x \right) \right\} \right)^{\frac{1}{3}} + \left( \sin \left\{ 2 \left( \frac{\pi}{4} - x \right) \right\} \right)^{\frac{1}{3}}} dx$$

$$\left\{ \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right\}$$

$$\text{So } I = \int_{\pi/24}^{5\pi/24} \frac{(\sin 2x)^{1/3}}{(\sin 2x)^{1/3} + (\cos 2x)^{1/3}} dx \quad \dots(ii)$$

$$\text{Hence } 2I = \int_{\pi/24}^{5\pi/24} dx \quad [(i) + (ii)]$$

$$\Rightarrow 2I = \frac{4\pi}{24} \Rightarrow I = \boxed{\frac{\pi}{12}}$$

## 6. Official Ans. by NTA (1)

$$\text{Sol. } f : [0, \infty) \rightarrow [0, \infty), f(x) = \int_0^x [y] dy$$

$$\text{Let } x = n + f, f \in (0, 1)$$

$$\text{So } f(x) = 0 + 1 + 2 + \dots + (n-1) + \int_n^{n+f} n dy$$

$$f(x) = \frac{n(n-1)}{2} + nf$$

$$= \frac{[x][[x]-1]}{2} + [x]\{x\}$$

$$\text{Note } \lim_{x \rightarrow n^+} f(x) = \frac{n(n-1)}{2},$$

$$\lim_{x \rightarrow n^-} f(x) = \frac{(n-1)(n-2)}{2} + (n-1) = \frac{n(n-1)}{2}$$

$$f(x) = \frac{n(n-1)}{2} \quad (n \in \mathbb{N}_0)$$

so  $f(x)$  is cont.  $\forall x \geq 0$  and diff. except at integer points

## 7. Official Ans. by NTA (3)

$$\begin{aligned}\text{Sol. } f(x) &= \int_0^1 (5 + (1-t)) dt + \int_1^x (5 + (t-1)) dt \\ &= 6 - \frac{1}{2} + \left( 4t + \frac{t^2}{2} \right) \Big|_1^x \\ &= \frac{11}{2} + 4x + \frac{x^2}{2} - 4 - \frac{1}{2} \\ &= \frac{x^2}{2} + 4x + 1\end{aligned}$$

$$f(2^+) = 2 + 8 + 1 = 11$$

$$f(2) = f(2^-) = 5 \times 2 + 1 = 11$$

$\Rightarrow$  continuous at  $x = 2$

Clearly differentiable at  $x = 1$

$$Lf'(2) = 5$$

$$Rf'(2) = 6$$

$\Rightarrow$  not differentiable at  $x = 2$

## 8. Official Ans. by NTA (2)

$$\begin{aligned}\text{Sol. Let } I &= \int_{-1}^1 \log\left(x + \sqrt{x^2 + 1}\right) dx \\ &\because \log\left(x + \sqrt{x^2 + 1}\right) \text{ is an odd function} \\ &\therefore I = 0\end{aligned}$$

## 9. Official Ans. by NTA (2)

$$\begin{aligned}\text{Sol. } I &= \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{x \cos x})(\sin^4 x + \cos^4 x)} \quad \dots(1) \\ \text{Using } \int_a^b f(x) dx &= \int_a^b f(a+b-x) dx \\ I &= \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{-x \cos x})(\sin^4 x + \cos^4 x)}\end{aligned}$$

Add (1) and (2)

$$2I = \int_{-\pi/4}^{\pi/4} \frac{dx}{\sin^4 x + \cos^4 x}$$

$$\begin{aligned}2I &= 2 \int_0^{\pi/4} \frac{dx}{\sin^4 x + \cos^4 x} \\ I &= \int_0^{\pi/4} \frac{(1 + \tan^2 x) \sec^2 x}{\tan^4 x + 1} dx \\ I &= \int_0^{\pi/4} \frac{\left(1 + \frac{1}{\tan^2 x}\right) \sec^2 x}{\left(\tan x - \frac{1}{\tan x}\right)^2 + 2} dx \\ \tan x - \frac{1}{\tan x} &= t \\ \left(1 + \frac{1}{\tan^2 x}\right) \sec^2 x dx &= dt\end{aligned}$$

$$\begin{aligned}I &= \int_{-\infty}^0 \frac{dt}{t^2 + 2} = \left[ \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) \right]_{-\infty}^0 \\ I &= 0 - \frac{1}{\sqrt{2}} \left(-\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}\end{aligned}$$

## 10. Official Ans. by NTA (1)

Sol. For domain

$$\log_5 (\log_3 (18x - x^2 - 77)) > 0$$

$$\log_3 (18x - x^2 - 77) > 1$$

$$18x - x^2 - 77 > 3$$

$$x^2 - 18x + 80 < 0$$

$$x \in (8, 10)$$

$$\Rightarrow a = 8 \text{ and } b = 10$$

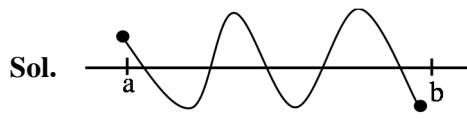
$$I = \int_a^b \frac{\sin^3 x}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$I = \int_a^b \frac{\sin^3(a+b-x)}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$2I = (b-a) \Rightarrow I = \frac{b-a}{2} \quad (\because a=8 \text{ and } b=10)$$

$$I = \frac{10-8}{2} = 1$$

## 11. Official Ans. by NTA (3)



$$f(x) = \int_a^x g(t) dt$$

$$f(x) \rightarrow 5$$

$$f'(x) \rightarrow 4$$

$$g(x) \rightarrow 4$$

$$g'(x) \rightarrow 3$$

## 12. Official Ans. by NTA (5)

$$\text{Sol. } I = 2 \int_0^{\pi/2} \sin^3 x e^{-\sin^2 x} dx$$

$$= 2 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx +$$

$$\int_0^{\pi/2} \underbrace{\cos x}_{\text{I}} \underbrace{e^{-\sin^2 x} (-\sin 2x)}_{\text{II}} dx$$

$$= 2 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx + \left[ \cos x e^{-\sin^2 x} \right]_0^{\pi/2}$$

$$+ \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx$$

$$= 3 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx - 1$$

$$= \frac{3}{2} \int_{-1}^0 \frac{e^\alpha d\alpha}{\sqrt{1+\alpha}} - 1 \quad (\text{Put } -\sin^2 x = t)$$

$$= \frac{3}{2} \int_0^1 \frac{e^x}{\sqrt{x}} dx - 1 \quad (\text{put } 1+\alpha=x)$$

$$= \frac{3}{2} \int_0^1 e^x \frac{1}{\sqrt{x}} dx - 1$$

$$= 2 - \frac{3}{2} \int_0^1 e^x \sqrt{x} dx$$

$$\text{Hence, } \alpha + \beta = \boxed{5}$$

## 13. Official Ans. by NTA (2)

$$\text{Sol. } I = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left( \left( \frac{x+1}{x-1} - \frac{x-1}{x+1} \right)^2 \right)^{1/2} dx$$

$$I = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left| \frac{4x}{x^2-1} \right| dx \Rightarrow I = 2.4 \int_0^{\sqrt{2}/2} \left| \frac{x}{x^2-1} \right| dx$$

$$\Rightarrow I = -4 \int_0^{\sqrt{2}/2} \frac{2x}{x^2-1} dx \Rightarrow I = -4 \ln|x^2-1| \Big|_0^{\sqrt{2}/2}$$

$$\Rightarrow I = 4 \ln 2 \Rightarrow I = \ln 16$$

## 14. Official Ans. by NTA (2)

$$\text{Sol. } L = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{r=0}^{2n-1} \frac{1}{1+4\left(\frac{r}{n}\right)^2}$$

$$\Rightarrow L = \int_0^2 \frac{1}{1+4x^2} dx$$

$$\Rightarrow L = \frac{1}{2} \tan^{-1}(2x) \Big|_0^2 \Rightarrow L = \frac{1}{2} \tan^{-1} 4$$

## 15. Official Ans. by NTA (2)

$$\text{Sol. } I = \int_0^5 \frac{x + [x]}{e^{x-[x]}} dx$$

$$\int_0^1 \frac{x}{e^x} dx + \int_1^2 \frac{x+1}{e^{x-1}} dx + \int_2^3 \frac{x+2}{e^{x-2}} dx + \dots + \int_4^5 \frac{x+4}{e^{x-4}} dx$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \\ x = t+1 \qquad x = z+2 \qquad x = y+4$$

$$\int_0^1 \frac{t+2}{e^t} dt + \int_0^2 \frac{z+4}{e^z} dz + \dots + \int_0^4 \frac{y+8}{e^y} dy$$

$$\Rightarrow \int_0^5 \frac{5x+20}{e^x} dx = 5 \int_0^1 \frac{x+4}{e^x} dx$$

$$\Rightarrow 5 \int_0^1 (x+4)e^{-x} dx$$

$$\Rightarrow 5e^{-x}(-x-5) \Big|_0^1 \Rightarrow -\frac{30}{e} + 25$$

$$\alpha = -30$$

$$\beta = 25 \Rightarrow 5\alpha + 6\beta = 0$$

$$(\alpha + \beta)^2 = 5^2 = 25$$

## 16. Official Ans. by NTA (3)

$$\text{Sol. } I = \int_0^{\pi/2} \frac{(1+\sin^2 x)}{(1+\pi^{\sin x})} + \frac{\pi^{\sin x} (1+\sin^2 x)}{(1+\pi^{\sin x})} dx$$

$$I = \int_0^{\pi/2} (1+\sin^2 x) dx$$

$$I = \frac{\pi}{2} + \frac{\pi}{2} \cdot \frac{1}{2} = \frac{3\pi}{4}$$

## 17. Official Ans. by NTA (1)

$$\text{Sol. } U_n = \prod_{r=1}^n \left(1 + \frac{r^2}{n^2}\right)^r$$

$$L = \lim_{n \rightarrow \infty} (U_n)^{-4/n^2}$$

$$\log L = \lim_{n \rightarrow \infty} \frac{-4}{n^2} \sum_{r=1}^n \log \left(1 + \frac{r^2}{n^2}\right)^r$$

$$\Rightarrow \log L = \lim_{n \rightarrow \infty} \sum_{r=1}^n -\frac{4r}{n} \cdot \frac{1}{n} \log \left(1 + \frac{r^2}{n^2}\right)$$

$$\Rightarrow \log L \Rightarrow -4 \int_0^1 x \log(1+x^2) dx$$

put  $1+x^2 = t$

Now,  $2x dx = dt$

$$= -2 \int_1^2 \log(t) dt = -2 [t \log t - t]_1^2$$

$$\Rightarrow \log L = -2(2 \log 2 - 1)$$

$$\therefore L = e^{-2(2 \log 2 - 1)}$$

$$= e^{-2 \left(\log \left(\frac{4}{e}\right)\right)}$$

$$= e^{\log \left(\frac{4}{e}\right)^{-2}}$$

$$= \left(\frac{e}{4}\right)^2 = \frac{e^2}{16}$$

## 18. Official Ans. by NTA (3)

$$\text{Sol. Let } I = \int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x^2 - 44x + 484)} dx$$

$$I = \int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x-22)^2} dx \quad \dots(1)$$

We know

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx \quad (\text{king})$$

So

$$I = \int_6^{16} \frac{\log_e (22-x)^2}{\log_e (22-x)^2 + \log_e (22-(22-x))^2} dx$$

$$I = \int_0^{16} \frac{\log_e (22-x)^2}{\log_e x^2 + \log_e (22-x)^2} dx \quad \dots(2)$$

(1) + (2)

$$2I = \int_6^{16} 1 dx = 10$$

$$I = 5$$

## 19. Official Ans. by NTA (1)

**Sol.**  $I = \int_0^1 \frac{\sqrt{x}}{(1+x)(1+3x)(3+x)} dx$

Let  $x = t^2 \Rightarrow dx = 2t dt$

$$I = \int_0^1 \frac{t(2t)}{(t^2+1)(1+3t^2)(3+t^2)} dt$$

$$I = \int_0^1 \frac{(3t^2+1)-(t^2+1)}{(3t^2+1)(t^2+1)(3+t^2)} dt$$

$$I = \int_0^1 \frac{dt}{(t^2+1)(3+t^2)} - \int_0^1 \frac{dt}{(1+3t^2)(3+t^2)}$$

$$= \frac{1}{2} \int_0^1 \frac{(3+t^2)-(t^2+1)}{(t^2+1)(3+t^2)} dt + \frac{1}{8} \int_0^1 \frac{(1+3t^2)-3(3+t^2)}{(1+3t^2)(3+t^2)} dt$$

$$= \frac{1}{2} \int_0^1 \frac{dt}{1+t^2} - \frac{1}{2} \int_0^1 \frac{dt}{t^2+3}$$

$$+ \frac{1}{8} \int_0^1 \frac{dt}{t^2+3} - \frac{3}{8} \int_0^1 \frac{dt}{1+3t^2}$$

$$= \frac{1}{2} \int_0^1 \frac{dt}{t^2+1} - \frac{3}{8} \int_0^1 \frac{dt}{t^2+3} - \frac{3}{8} \int_0^1 \frac{dt}{1+3t^2}$$

$$= \frac{1}{2} \left( \tan^{-1}(t) \right)_0^1 - \frac{3}{8\sqrt{3}} \left( \tan^{-1} \left( \frac{t}{\sqrt{3}} \right) \right)_0^1$$

$$- \frac{3}{8\sqrt{3}} \left( \tan^{-1}(\sqrt{3}t) \right)_0^1$$

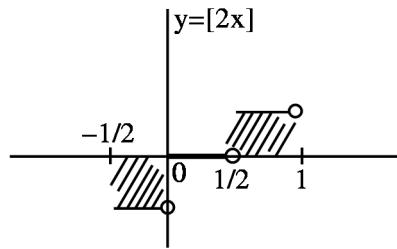
$$= \frac{1}{2} \left( \frac{\pi}{4} \right) - \frac{\sqrt{3}}{8} \left( \frac{\pi}{6} \right) - \frac{\sqrt{3}}{8} \left( \frac{\pi}{3} \right)$$

$$= \frac{\pi}{8} - \frac{\sqrt{3}}{16} \pi$$

$$= \frac{\pi}{8} \left( 1 - \frac{\sqrt{3}}{2} \right)$$

## 20. Official Ans. by NTA (5)

**Sol.**  $I = \int_{-1/2}^1 ([2x] + |x|) dx$



$$= \int_{-1/2}^1 [2x] dx + \int_{-1/2}^1 |x| dx$$

$$= 0 + \int_{-1/2}^0 (-x) dx + \int_0^1 x dx$$

$$= \left( -\frac{x^2}{2} \right)_{-1/2}^0 + \left( \frac{x^2}{2} \right)_0^1$$

$$= \left( 0 + \frac{1}{8} \right) + \frac{1}{2}$$

$$= \frac{5}{8}$$

$$8I = 5$$

## 21. Official Ans. by NTA (4)

**Sol.**  $x\phi(x) = \int_5^x 3t^2 - 2\phi'(t) dt$

$$x\phi(x) = x^3 - 125 - 2[\phi(x) - \phi(5)]$$

$$x\phi(x) = x^3 - 125 - 2\phi(x) - 2\phi(5)$$

$$\phi(0) = 4 \Rightarrow \phi(5) = -\frac{133}{2}$$

$$\phi(x) = \frac{x^3 + 8}{x + 2}$$

$$\phi(2) = 4$$

## 22. Official Ans. by NTA (2)

**Sol.**  $\pi^2 \left[ \int_0^1 \sin \frac{\pi x}{2} dx + \int_1^2 \sin \frac{\pi x}{2} (x-1) dx \right]$

$$= \pi^2 \left[ -\frac{2}{\pi} \left( \cos \frac{\pi x}{2} \right) + \left( (x-1) \left( -\frac{2}{\pi} \cos \frac{\pi x}{2} \right) \right)_1^2 - \int_1^2 -\frac{2}{\pi} \cos \frac{\pi x}{2} dx \right]$$

$$= \pi^2 \left[ 0 + \frac{2}{\pi} + \frac{2}{\pi} + \frac{2}{\pi} \cdot \frac{2}{\pi} \left( \sin \frac{\pi x}{2} \right)_1^2 \right]$$

$$= 4\pi - 4 = 4(\pi - 1)$$

## 23. Official Ans. by NTA (4)

Sol.  $f(x) = x + \int_0^{\pi/2} \sin x \cos y f(y) dy$

$$f(x) = x + \sin x \underbrace{\int_0^{\pi/2} \cos y f(y) dy}_K$$

$$\Rightarrow f(x) = x + K \sin x$$

$$\Rightarrow f(y) = y + K \sin y$$

$$\text{Now } K = \int_0^{\pi/2} \cos y (y + K \sin y) dy$$

$$K = \int_0^{\pi/2} y \cos y dy + \int_0^{\pi/2} \cos y \sin y dy$$

Apply IBP      Put  $\sin y = t$

$$K = (y \sin y) \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin y dy + K \int_0^1 t dt$$

$$\Rightarrow K = \frac{\pi}{2} - 1 + K \left( \frac{1}{2} \right)$$

$$\Rightarrow K = \pi - 2$$

$$\text{So } f(x) = x + (\pi - 2) \sin x$$

Option (4)

## 24. Official Ans. by NTA (2)

Sol.  $\int_1^3 \left[ (x-1)^2 \right] - 3 dx$

$$= \int_1^2 [x^2] - 3 \int_1^3 dx$$

$$= \int_1^3 0 dx + \int_1^2 1 dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx - 6$$

$$= \sqrt{2} - 1 + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) - 6$$

$$= -\sqrt{2} - \sqrt{3} - 1$$

## 25. Official Ans. by NTA (2)

Sol.  $f'(x) = f'(2-x)$

$$f(x) = -f(2-x) + c$$

put  $x = 0$

$$f'(0) = -f'(2) + c$$

$$c = f(0) + f(2) = 1 + e^2$$

$$\text{so, } f(x) + f(2-x) = 1 + e^2$$

$$I = \int_0^2 f(x) dx$$

$$I = \int_0^2 f(2-x) dx$$

$$2I = \int_0^2 (f(x) + f(2-x)) dx$$

$$2I = (1+e^2) \int_0^2 dx$$

$$I = 1 + e^2$$

## 26. Official Ans. by NTA (3)

Sol.  $\int_{-a}^0 (-2x+2) dx + \int_0^2 (x+2-x) dx + \int_2^a (2x-2) dx = 22$

$$x^2 - 2x \Big|_0^{-a} + 2x \Big|_0^2 + x^2 - 2x \Big|_0^a = 22$$

$$a^2 + 2a + 4 + a^2 - 2a - (4 - 4) = 22$$

$$2a^2 = 18 \Rightarrow a = 3$$

$$\int_3^{-3} (x + [x]) dx = -(-3 - 2 - 1 + 1 + 2) = 3$$

## 27. Official Ans. by NTA (3)

Sol.  $I = \int_{-1}^1 x^2 e^{[x^3]} dx$

$$= \int_{-1}^0 x^2 e^{[x^3]} dx + \int_0^1 x^2 e^{[x^3]} dx$$

$$= \int_{-1}^0 x^2 e^{-1} dx + \int_0^1 x^2 e^0 dx$$

$$= \frac{1}{e} \times \frac{x^3}{3} \Big|_{-1}^0 + \frac{x^3}{3} \Big|_0^1$$

$$= \frac{1}{e} \times \left( 0 - \left( \frac{-1}{3} \right) \right) + \frac{1}{3}$$

$$= \frac{1}{3e} + \frac{1}{3} = \frac{1+e}{3e}$$

## 28. Official Ans. by NTA (1)

$$\begin{aligned} \text{Sol. } & \lim_{n \rightarrow \infty} \left[ \frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \dots + \frac{n}{(2n-1)^2} \right] \\ &= \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{n}{(n+r)^2} = \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{n}{n^2 + 2nr + r^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{\left(\frac{r}{n}\right)^2 + 2\left(\frac{r}{n}\right) + 1} \\ &= \int_0^1 \frac{dx}{(x+1)^2} = \left[ \frac{-1}{(x+1)} \right]_0^1 = \frac{1}{2} \end{aligned}$$

## 29. Official Ans. by NTA (19)

$$\begin{aligned} \text{Sol. } & \int_{-2}^2 3|x^2 - x - 2| dx \\ &= 3 \int_{-2}^2 |x^2 - x - 2| dx \\ &= 3 \left[ \int_{-2}^{-1} (x^2 - x - 2) dx + \int_{-1}^2 -(x^2 - x - 2) dx \right] \\ &= 3 \left[ \left( \frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \Big|_{-2}^{-1} - \left( \frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \Big|_{-1}^2 \right] \\ &= 3 \left[ 7 - \frac{2}{3} \right] \\ &= 19 \end{aligned}$$

## 30. Official Ans. by NTA (3)

$$\begin{aligned} \text{Sol. } & f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt \\ & f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ell n t}{1+t} dt, \text{ let } t = \frac{1}{y} \\ &= + \int_1^x \frac{\ell n y}{1+y} \cdot \frac{y}{y^2} dy \\ &= \int_1^x \frac{\ell n y}{y(1+y)} dy \\ \text{hence } & f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{(1+t)\ell n t}{t(1+t)} dt = \int_1^x \frac{\ell n t}{t} dt \\ &= \frac{1}{2} \ell n^2(x) \end{aligned}$$

$$\text{so } f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2} \quad \dots(3)$$

## 31. Official Ans. by NTA (1)

$$\text{Sol. } f(x) = \int_0^x e^t f(t) dt + e^x \Rightarrow f(0) = 1$$

differentiating with respect to x

$$f'(x) = e^x f(x) + e^x$$

$$f'(x) = e^x(f(x) + 1)$$

$$\int_0^x \frac{f'(x)}{f(x)+1} dx = \int_0^x e^x dx$$

$$\ell n(f(x)+1) \Big|_0^x = e^x \Big|_0^x$$

$$\ell n(f(x)+1) - \ell n(f(0)+1) = e^x - 1$$

$$\ell n\left(\frac{f(x)+1}{2}\right) = e^x - 1 \quad \{ \text{as } f(0) = 1 \}$$

$$f(x) = 2e^{(e^x-1)} - 1$$

## 32. Official Ans. by NTA (1)

$$\text{Sol. } I_{m,n} = \int_0^1 x^{m-1} (1-x)^{n-1} dx = I_{n,m}$$

$$\text{Now Let } x = \frac{1}{y+1} \Rightarrow dx = -\frac{1}{(y+1)^2} dy$$

so

$$I_{m,n} = - \int_{\infty}^0 \frac{1}{(y+1)^{m-1}} \frac{y^{n-1}}{(y+1)^{n-1}} \frac{dy}{(y+1)^2} = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\text{similarly } I_{m,n} = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$\text{Now } 2I_{m,n} = \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy + \underbrace{\int_1^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy}_{\text{substitute } y = \frac{1}{t}}$$

$$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy - \int_1^0 \frac{t^{n-1} + t^{m-1}}{t^{m+n-2}} \frac{t^{m+n}}{(1+t)^{m+n}} \frac{dt}{t^2}$$

$$\Rightarrow \text{Hence } 2I_{m,n} = 2 \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy \Rightarrow \alpha = 1$$

**33. Official Ans. by NTA (2)**

**Sol.** Put  $2x = t \Rightarrow 2dx = dt$

$$\Rightarrow I = \frac{1}{2} \int_0^{2\pi} |\sin t| dt$$

$$= \int_0^{\pi} |\sin t| dt \\ = 2$$

**34. Official Ans. by NTA (1)**

$$\text{Sol. } I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} dx \text{ (using king)}$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^{-x}} dx = \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{1+3^x} dx$$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{(1+3^x) \cos^2 x}{1+3^x} dx \\ = \int_{-\pi/2}^{\pi/2} \cos^2 x dx = 2 \int_0^{\pi/2} \cos^2 x dx$$

$$\Rightarrow I = \int_0^{\pi/2} \cos^2 x dx = \frac{\pi}{4}$$

**35. Official Ans. by NTA (4)**

$$\text{Sol. } I_n = \int_{\pi/4}^{\pi/2} \cot^n x dx = \int_{\pi/4}^{\pi/2} \cot^{n-2} x (\cosec^2 x - 1) dx$$

$$= -\frac{\cot^{n-1} x}{n-1} \Big|_{\pi/4}^{\pi/2} - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

$$\Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$$

$$\Rightarrow I_2 + I_4 = \frac{1}{3}$$

$$I_3 + I_5 = \frac{1}{4}$$

$$I_4 + I_6 = \frac{1}{5}$$

$\therefore \frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}$  are in A.P.

**36. Official Ans. by NTA (1)**

$$\text{Sol. } \sum_{n=1}^{100} \int_{n-1}^n e^{\{x\}} dx, \text{ period of } \{x\} = 1$$

$$\sum_{n=1}^{100} \int_0^1 e^{\{x\}} dx = \sum_{n=1}^{100} \int_0^1 e^x dx$$

$$\sum_{n=1}^{100} (e-1) = 100(e-1)$$

**37. Official Ans by NTA (3)**

$$\text{Sol. } I = \int_0^{10} [x] \cdot e^{[x]-x+1}$$

$$I = \int_0^1 0 dx + \int_1^2 1 \cdot e^{2-x} + \int_2^3 2 \cdot e^{3-x} + \dots + \int_9^{10} 9 \cdot e^{10-x} dx$$

$$\Rightarrow I = \sum_{n=0}^9 \int_n^{n+1} n \cdot e^{n+1-x} dx$$

$$= - \sum_{n=0}^9 n (e^{n+1-x})_n^{n+1}$$

$$= - \sum_{n=0}^9 n \cdot (e^0 - e^1)$$

$$= (e-1) \sum_{n=0}^9 n$$

$$= (e-1) \cdot \frac{9 \cdot 10}{2} \\ = 45(e-1)$$

**38. Official Ans by NTA (3)**

$$\text{Sol. } \int_0^1 (x^2 + bx + c) dx = 1$$

$$\frac{1}{3} + \frac{b}{2} + c = 1 \Rightarrow \frac{b}{2} + c = \frac{2}{3}$$

$$3b + 6c = 4 \quad \dots(1)$$

$$P(2) = 5$$

$$4 + 2b + c = 5$$

$$2b + c = 1 \quad \dots(2)$$

From (1) & (2)

$$b = \frac{2}{9} \quad \& \quad c = \frac{5}{9}$$

$$9(b+c) = 7$$

## 39. Official Ans. by NTA (16)

**Sol.**  $f(x) + f(x+1) = 2$

$\Rightarrow f(x)$  is periodic with period = 2

$$I_1 = \int_0^8 f(x) dx = 4 \int_0^2 f(x) dx$$

$$= 4 \int_0^1 (f(x) + f(1+x)) dx = 8$$

Similarly  $I_2 = 2 \times 2 = 4$

$$I_1 + 2I_2 = 16$$

## 40. Official Ans. by NTA (2)

**Sol.**  $f(x) = e^{-x} \sin x$

Now,  $F(x) = \int_0^x f(t) dt \Rightarrow F'(x) = f(x)$

$$I = \int_0^1 (F'(x) + f(x)) e^x dx = \int_0^1 (f(x) + f(x)) \cdot e^x dx$$

$$= 2 \int_0^1 f(x) \cdot e^x dx = 2 \int_0^1 e^{-x} \sin x \cdot e^x dx$$

$$= 2 \int_0^1 \sin x dx$$

$$= 2(1 - \cos 1)$$

$$I = 2 \left\{ 1 - \left( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \dots \right) \right\}$$

$$I = 1 - \frac{2}{4} + \frac{2}{6} - \frac{2}{9} + \dots$$

$$1 - \frac{2}{4} < I < 1 - \frac{2}{4} + \frac{2}{6}$$

$$\frac{11}{12} < I < \frac{331}{360}$$

$$\Rightarrow I \in \left[ \frac{11}{12}, \frac{331}{360} \right]$$

$$\Rightarrow I \in \left[ \frac{330}{360}, \frac{331}{360} \right] \quad \text{Ans. (2)}$$

## 41. Official Ans. by NTA (1)

**Sol.** Let  $I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \int_0^{10} \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$

Function  $f(x) = \frac{[\sin 2\pi x]}{e^{\{x\}}}$  is periodic with period '1'

Therefore

$$I = 10 \int_0^1 \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$$

$$= 10 \int_0^1 \frac{[\sin 2\pi x]}{e^x} dx$$

$$= 10 \left( \int_0^{1/2} \frac{[\sin 2\pi x]}{e^x} dx + \int_{1/2}^1 \frac{[\sin 2\pi x]}{e^x} dx \right)$$

$$= 10 \left( 0 + \int_{1/2}^1 \frac{(-1)}{e^x} dx \right)$$

$$= -10 \int_{1/2}^1 e^{-x} dx$$

$$= 10(e^{-1} - e^{-1/2})$$

Now,

$$10 \cdot e^{-1} - 10 \cdot e^{-1/2} = \alpha e^{-1} + \beta e^{-1/2} + \gamma \text{ (given)}$$

$$\Rightarrow \alpha = 10, \beta = -10, \gamma = 0$$

$$\Rightarrow \alpha + \beta + \gamma = 0$$

**Ans. (1)**

## 42. Official Ans. by NTA (1)

**Sol.**  $I_n = \int_1^e x^{19} (\log|x|)^n dx$

$$I_n = \left| \left( \log|x| \right)^{19} \frac{x^{20}}{20} \right|_1^e - \int n(\log|x|)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{20}}{20} dx$$

$$20I_n = e^{20} - nI_{n-1}$$

$$\therefore 20I_{10} = e^{20} - 10I_9$$

$$20I_9 = e^{20} - 9I_8$$

$$\Rightarrow 20I_{10} = 10I_9 + 9I_8$$

$$\alpha = 10, \beta = 9$$

## 43. Official Ans. by NTA (Bonus)

Sol. 
$$g(\alpha) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^\alpha x}{(\sin^\alpha x + \cos^\alpha x)} \dots \text{(i)}$$

$$g(\alpha) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^\alpha x}{(\sin^\alpha x + \cos^\alpha x)} \dots \text{(ii)}$$

$$(1) + (2)$$

$$2g(\alpha) = \frac{\pi}{6}$$

$$g(\alpha) = \frac{\pi}{12}$$

Constant and even function

Due to typing mistake it must be bonus.

## 44. Official Ans. by NTA (512)

Sol.  $I = 2 \int_0^4 f(x^2) dx \quad \{ \text{Even function} \}$

$$= 2 \int_0^4 (4x^3 - g(4-x)) dx$$

$$= 2 \left( \frac{4x^4}{4} \Big|_0^4 - \int_0^4 g(4-x) dx \right)$$

$$= 2(256 - 0) = 512$$

## 45. Official Ans. by NTA (3)

Sol.  $\frac{1}{3} \leq f(t) \leq 1 \quad \forall t \in [0, 1]$

$$0 \leq f(t) \leq \frac{1}{2} \quad \forall t \in (1, 3]$$

$$\text{Now, } g(3) = \int_0^3 f(t) dt = \int_0^1 f(t) dt + \int_1^3 f(t) dt$$

$$\therefore \int_0^1 \frac{1}{3} dt \leq \int_0^1 f(t) dt \leq \int_0^1 1 dt \quad \dots \text{(1)}$$

$$\text{and } \int_1^3 0 dt \leq \int_1^3 f(t) dt \leq \int_1^3 \frac{1}{2} dt \quad \dots \text{(2)}$$

Adding, we get

$$\frac{1}{3} + 0 \leq g(3) \leq 1 + \frac{1}{2}(3-1)$$

$$\frac{1}{3} \leq g(3) \leq 2$$