

COMPLEX NUMBER

- If z and ω are two complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \frac{3\pi}{2}$, then $\arg\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right)$ is : (Here $\arg(z)$ denotes the principal argument of complex number z)
 (1) $\frac{\pi}{4}$ (2) $-\frac{3\pi}{4}$ (3) $-\frac{\pi}{4}$ (4) $\frac{3\pi}{4}$
- If the real part of the complex number $(1 - \cos\theta + 2i\sin\theta)^{-1}$ is $\frac{1}{5}$ for $\theta \in (0, \pi)$, then the value of the integral $\int_0^\theta \sin x \, dx$ is equal to :
 (1) 1 (2) 2 (3) -1 (4) 0
- Let n denote the number of solutions of the equation $z^2 + 3\bar{z} = 0$, where z is a complex number. Then the value of $\sum_{k=0}^{\infty} \frac{1}{n^k}$ is equal to
 (1) 1 (2) $\frac{4}{3}$ (3) $\frac{3}{2}$ (4) 2
- Let $S = \left\{ n \in \mathbf{N} \left| \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \forall a, b, c, d \in \mathbf{R} \right\}$, where $i = \sqrt{-1}$. Then the number of 2-digit numbers in the set S is _____.
- The equation of a circle is $\text{Re}(z^2) + 2(\text{Im}(z))^2 + 2\text{Re}(z) = 0$, where $z = x + iy$. A line which passes through the center of the given circle and the vertex of the parabola, $x^2 - 6x - y + 13 = 0$, has y-intercept equal to _____.
- Let C be the set of all complex numbers. Let $S_1 = \{z \in C \mid |z - 3 - 2i|^2 = 8\}$, $S_2 = \{z \in C \mid \text{Re}(z) \geq 5\}$ and $S_3 = \{z \in C \mid |z - \bar{z}| \geq 8\}$. Then the number of elements in $S_1 \cap S_2 \cap S_3$ is equal to
 (1) 1 (2) 0
 (3) 2 (4) Infinite

- Let $\mathbb{C}$ be the set of all complex numbers. Let $S_1 = \{z \in \mathbb{C} : |z - 2| \leq 1\}$ and $S_2 = \{z \in \mathbb{C} : z(1+i) + \bar{z}(1-i) \geq 4\}$. Then, the maximum value of $\left|z - \frac{5}{2}\right|^2$ for $z \in S_1 \cap S_2$ is equal to :
 (1) $\frac{3+2\sqrt{2}}{4}$ (2) $\frac{5+2\sqrt{2}}{2}$
 (3) $\frac{3+2\sqrt{2}}{2}$ (4) $\frac{5+2\sqrt{2}}{4}$
- If the real part of the complex number $z = \frac{3+2i\cos\theta}{1-3i\cos\theta}$, $\theta \in \left(0, \frac{\pi}{2}\right)$ is zero, then the value of $\sin^2 3\theta + \cos^2 \theta$ is equal to _____.
- The equation $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ represents a circle with:
 (1) centre at $(0, -1)$ and radius $\sqrt{2}$
 (2) centre at $(0, 1)$ and radius $\sqrt{2}$
 (3) centre at $(0, 0)$ and radius $\sqrt{2}$
 (4) centre at $(0, 1)$ and radius 2
- Let $z = \frac{1-i\sqrt{3}}{2}$, $i = \sqrt{-1}$. Then the value of $21 + \left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \left(z^3 + \frac{1}{z^3}\right)^3 + \dots + \left(z^{21} + \frac{1}{z^{21}}\right)^3$ is _____.
- If $(\sqrt{3} + i)^{100} = 2^{99}(p + iq)$, then p and q are roots of the equation :
 (1) $x^2 - (\sqrt{3} - 1)x - \sqrt{3} = 0$
 (2) $x^2 + (\sqrt{3} + 1)x + \sqrt{3} = 0$
 (3) $x^2 + (\sqrt{3} - 1)x - \sqrt{3} = 0$
 (4) $x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$

12. The least positive integer n such that $\frac{(2i)^n}{(1-i)^{n-2}}, i = \sqrt{-1}$ is a positive integer, is _____.
13. If $S = \left\{ z \in \mathbb{C} : \frac{z-i}{z+2i} \in \mathbb{R} \right\}$, then :
 (1) S contains exactly two elements
 (2) S contains only one element
 (3) S is a circle in the complex plane
 (4) S is a straight line in the complex plane
14. Let z_1 and z_2 be two complex numbers such that $\arg(z_1 - z_2) = \frac{\pi}{4}$ and z_1, z_2 satisfy the equation $|z - 3| = \operatorname{Re}(z)$. Then the imaginary part of $z_1 + z_2$ is equal to _____.
15. A point z moves in the complex plane such that $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$, then the minimum value of $|z - 9\sqrt{2} - 2i|^2$ is equal to _____.
16. If z is a complex number such that $\frac{z-i}{z-1}$ is purely imaginary, then the minimum value of $|z - (3 + 3i)|$ is :
 (1) $2\sqrt{2} - 1$ (2) $3\sqrt{2}$
 (3) $6\sqrt{2}$ (4) $2\sqrt{2}$
17. If for the complex numbers z satisfying $|z - 2 - 2i| \leq 1$, the maximum value of $|3iz + 6|$ is attained at $a + ib$, then $a + b$ is equal to _____.
18. Let $i = \sqrt{-1}$. If $\frac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}} + \frac{(1+i\sqrt{3})^{21}}{(1+i)^{24}} = k$, and $n = [k]$ be the greatest integral part of $|k|$.
 Then $\sum_{j=0}^{n+5} (j+5)^2 - \sum_{j=0}^{n+5} (j+5)$ is equal to _____.
19. If the least and the largest real values of α , for which the equation $z + \alpha|z - 1| + 2i = 0$ ($z \in \mathbb{C}$ and $i = \sqrt{-1}$) has a solution, are p and q respectively; then $4(p^2 + q^2)$ is equal to _____.
20. Let the lines $(2 - i)z = (2 + i)\bar{z}$ and $(2 + i)z + (i - 2)\bar{z} - 4i = 0$, (here $i^2 = -1$) be normal to a circle C . If the line $iz + \bar{z} + 1 + i = 0$ is tangent to this circle C , then its radius is:
 (1) $\frac{3}{\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$ (3) $3\sqrt{2}$ (4) $\frac{3}{2\sqrt{2}}$
21. Let z be those complex numbers which satisfy $|z + 5| \leq 4$ and $z(1+i) + \bar{z}(1-i) \geq -10, i = \sqrt{-1}$. If the maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$, then the value of $(\alpha + \beta)$ is _____.
22. The sum of 162^{th} power of the roots of the equation $x^3 - 2x^2 + 2x - 1 = 0$ is
23. The least value of $|z|$ where z is complex number which satisfies the inequality $\exp\left(\frac{(|z|+3)(|z|-1)}{|z|+1} \log_e 2\right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|$, $i = \sqrt{-1}$, is equal to :
 (1) 3 (2) $\sqrt{5}$ (3) 2 (4) 8
24. Let a complex number $z, |z| \neq 1$, satisfy $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2$. Then, the largest value of $|z|$ is equal to _____.
 (1) 8 (2) 7 (3) 6 (4) 5
25. Let $P = \begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 60 & 14 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega + 1 \end{bmatrix}$
 Where $\omega = \frac{-1+i\sqrt{3}}{2}$, and I_3 be the identity matrix of order 3. If the determinant of the matrix $(P^{-1}AP - I_3)^2$ is $\alpha\omega^2$, then the value of α is equal to _____.

26. Let z and w be two complex numbers such that $w = z\bar{z} - 2z + 2$, $\left| \frac{z+i}{z-3i} \right| = 1$ and $\operatorname{Re}(w)$ has minimum value. Then, the minimum value of $n \in \mathbb{N}$ for which w^n is real, is equal to _____.

27. Let S_1, S_2 and S_3 be three sets defined as

$$S_1 = \{z \in \mathbb{C} : |z-1| \leq \sqrt{2}\}$$

$$S_2 = \{z \in \mathbb{C} : \operatorname{Re}((1-i)z) \geq 1\}$$

$$S_3 = \{z \in \mathbb{C} : \operatorname{Im}(z) \leq 1\}$$

Then the set $S_1 \cap S_2 \cap S_3$

- (1) is a singleton
- (2) has exactly two elements
- (3) has infinitely many elements
- (4) has exactly three elements

28. The area of the triangle with vertices $A(z)$, $B(iz)$ and $C(z+iz)$ is :

- (1) 1
- (2) $\frac{1}{2}|z|^2$
- (3) $\frac{1}{2}$
- (4) $\frac{1}{2}|z+iz|^2$

29. If the equation $a|z|^2 + \bar{\alpha}z + \alpha\bar{z} + d = 0$ represents a circle where a, d are real constants then which of the following condition is correct?

- (1) $|\alpha|^2 - ad \neq 0$
- (2) $|\alpha|^2 - ad > 0$ and $a \in \mathbb{R} - \{0\}$
- (3) $|\alpha|^2 - ad \geq 0$ and $a \in \mathbb{R}$
- (4) $\alpha = 0, a, d \in \mathbb{R}^+$

30. Let z_1, z_2 be the roots of the equation $z^2 + az + 12 = 0$ and z_1, z_2 form an equilateral triangle with origin. Then, the value of $|a|$ is

31. Let a complex number be $w = 1 - \sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w is equal to :

- (1) 4
- (2) $\frac{1}{2}$
- (3) $\frac{1}{4}$
- (4) 2

32. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $P(x) = f(x^3) + xg(x^3)$ is divisible by $x^2 + x + 1$, then $P(1)$ is equal to _____.

SOLUTION

1. Official Ans. by NTA (3)

ALLEN Ans. (2)

Sol. As $|z\omega| = 1$

$$\Rightarrow \text{If } |z| = r, \text{ then } |\omega| = \frac{1}{r}$$

Let $\arg (z)=\theta$

$$\therefore \arg (\omega) = \left(\theta - \frac{3\pi}{2} \right)$$

So, $z = re^{i\theta}$

$$\Rightarrow \bar{z} = re^{i(-\theta)}$$

$$\omega = \frac{1}{r} e^{i\left(\theta - \frac{3\pi}{2}\right)}$$

Now, consider

$$\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega} = \frac{1-2e^{i\left(-\frac{3\pi}{2}\right)}}{1+3e^{i\left(-\frac{3\pi}{2}\right)}} = \left(\frac{1-2i}{1+3i}\right)$$

$$= \frac{(1-2i)(1-3i)}{(1+3i)(1-3i)} = -\frac{1}{2}(1+i)$$

$$\therefore \text{prin arg} \left(\frac{1 - 2\bar{z}\omega}{1 + 3\bar{z}\omega} \right)$$

$$= \text{prin arg} \left(\frac{1 - 2\bar{z}\omega}{1 + 3\bar{z}\omega} \right)$$

$$= \left(-\frac{1}{2}(1 + i) \right)$$

$$= -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$$

So, option (2) is correct.

2. Official Ans. by NTA (1)

Sol. $z = \frac{1}{1 - \cos \theta + 2i \sin \theta}$

$$= \frac{2\sin^2 \frac{\theta}{2} - 2i\sin \theta}{(1 - \cos \theta)^2 + 4\sin^2 \theta}$$

$$= \frac{\sin \frac{\theta}{2} - 2i \cos \frac{\theta}{2}}{4 \sin \frac{\theta}{2} \left(\sin^2 \frac{\theta}{2} + 4 \cos^2 \frac{\theta}{2} \right)}$$

$$\operatorname{Re}(z) = \frac{1}{2\left(\sin^2 \frac{\theta}{2} + 4\cos^2 \frac{\theta}{2}\right)} = \frac{1}{5}$$

$$\sin \frac{2\theta}{2} + 4 \cos^2 \frac{\theta}{2} = \frac{5}{2}$$

$$1 - \cos^2 \frac{\theta}{2} + 4 \cos \frac{\theta}{2} = \frac{5}{2}$$

$$3\cos^2\frac{\theta}{2} = \frac{3}{2}$$

$$\cos^2 \frac{\theta}{2} = \frac{1}{2}$$

$$\frac{\theta}{2} = n\pi \pm \frac{\pi}{4}$$

$$\theta = 2n\pi \pm \frac{\pi}{2}$$

$$\theta = 2n\pi \pm \frac{\pi}{2}$$

$$\theta \in (0, \pi)$$

$\theta = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \sin \theta \, d\theta = [-\cos \theta]_0^{\frac{\pi}{2}}$$

$$= -(0 - 1)$$

$$= 1$$

3. Official Ans. by NTA (2)

Sol. $z^2 + 3\bar{z} = 0$

Put $z = x + iy$

$$\Rightarrow x^2 - y^2 + 2ixy + 3(x - iy) = 0$$

$$\Rightarrow (x^2 - y^2 + 3x) + i(2xy - 3y) = 0 + i0$$

$$\therefore x^2 - y^2 + 3x = 0 \quad \dots(1)$$

$$2xy - 3y = 0 \quad \dots(2)$$

$$x = \frac{3}{2}, y = 0$$

Put $x = \frac{3}{2}$ in equation (1)

$$\frac{9}{4} - y^2 + \frac{9}{2} = 0$$

$$y^2 = \frac{27}{4} \Rightarrow y = \pm \frac{3\sqrt{3}}{2}$$

$$\therefore (x, y) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right), \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$$

Put $y = 0 \Rightarrow x^2 - 0 + 3x = 0$

$$x = 0, -3$$

$$\therefore (x, y) = (0, 0), (-3, 0)$$

$$\therefore \text{No of solutions} = n = 4$$

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{1}{n^k}\right) &= \sum_{k=0}^{\infty} \left(\frac{1}{4^k}\right) \\ &= \frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots \\ &= \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \end{aligned}$$

4. Official Ans. by NTA (11)

Sol. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ & $A = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}^n$

$$\Rightarrow AX = IX$$

$$\Rightarrow A = I$$

$$\Rightarrow \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}^n = I$$

$$\Rightarrow A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow n \text{ is multiple of } 8$$

So number of 2 digit numbers in the set

$$S = 11 (16, 24, 32, \dots, 96)$$

5. Official Ans. by NTA (1)

Sol. Equation of circle is $(x^2 - y^2) + 2y^2 + 2x = 0$

$$x^2 + y^2 + 2x = 0$$

Centre : $(-1, 0)$

Parabola : $x^2 - 6x - y + 13 = 0$

$$(x - 3)^2 = y - 4$$

Vertex : $(3, 4)$

Equation of line $\equiv y - 0 = \frac{4-0}{3+1}(x+1)$

$$y = x + 1$$

y-intercept = 1

6. Official Ans. by NTA (1)

Sol. $S_1 : |z - 3 - 2i|^2 = 8$

$$|z - 3 - 2i| = 2\sqrt{2}$$

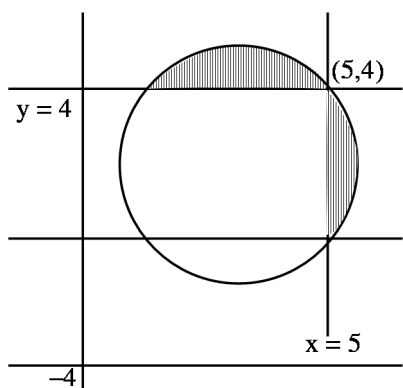
$$(x - 3)^2 + (y - 2)^2 = (2\sqrt{2})^2$$

$$S_2 : x \geq 5$$

$$S_3 : |z - \bar{z}| \geq 8$$

$$|2iy| \geq 8$$

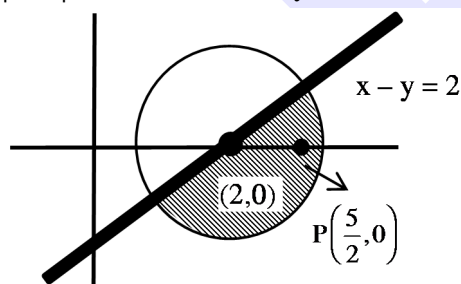
$$2|y| \geq 8 \therefore y \geq 4, y \leq -4$$



$$n(S_1 \cap S_2 \cap S_3) = 1$$

7. Official Ans. by NTA (4)

Sol. $|t - 2| \leq 1$ Put $t = x + iy$



$$(x - 2)^2 + y^2 \leq 1$$

$$\text{Also, } t(1 + i) + \bar{t}(1 - i) \geq 4$$

$$\text{Gives } x - y \geq 2$$

$$\text{Let point on circle be } A(2 + \cos \theta, \sin \theta)$$

$$\theta \in \left[-\frac{3\pi}{4}, \frac{\pi}{4} \right]$$

$$(AP)^2 = \left(2 + \cos \theta - \frac{5}{2} \right)^2 + \sin^2 \theta$$

$$= \cos^2 \theta - \cos \theta + \frac{1}{4} + \sin^2 \theta$$

$$= \frac{5}{4} - \cos \theta$$

$$\text{For } (AP)^2 \text{ maximum } \theta = -\frac{3\pi}{4}$$

$$(AP)^2 = \frac{5}{4} + \frac{1}{\sqrt{2}} = \frac{5\sqrt{2} + 4}{4\sqrt{2}}$$

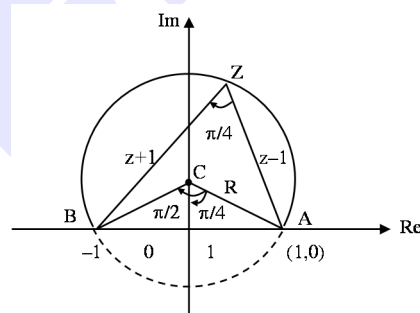
8. Official Ans. by NTA (1)

Sol. $\operatorname{Re}(z) = \frac{3 - 6\cos^2 \theta}{1 + 9\cos^2 \theta} = 0$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\text{Hence, } \sin^2 3\theta + \cos^2 \theta = 1.$$

9. Official Ans. by NTA (2)



Sol.

In $\triangle OAC$

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{AC}$$

$$\Rightarrow AC = \sqrt{2}$$

$$\text{Also, } \tan \frac{\pi}{4} = \frac{OA}{OC} = \frac{1}{OC}$$

$$\Rightarrow OC = 1$$

$$\therefore \text{centre } (0, 1); \text{ Radius} = \sqrt{2}$$

10. Official Ans. by NTA (13)

Sol. $Z = \frac{1 - \sqrt{3}i}{2} = e^{-i\frac{\pi}{3}}$

$$z^r + \frac{1}{z^r} = 2 \cos\left(-\frac{\pi}{3}\right) = 2 \cos \frac{r\pi}{3}$$

$$\Rightarrow 21 + \sum_{r=1}^{21} \left(z^r + \frac{1}{z^r}\right)^3 = 8 \left(\cos^3 \frac{r\pi}{3}\right) = 2 \left(\cos r\pi + 3 \cos \frac{r\pi}{3}\right)$$

$$\Rightarrow 21 + \left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \dots + \left(z^{21} + \frac{1}{z^{21}}\right)^3$$

$$= 21 + \sum_{r=1}^{21} \left(z^r + \frac{1}{z^r}\right)^3$$

$$= 21 + \sum_{r=1}^{21} \left(2 \cos r\pi + 6 \cos \frac{r\pi}{3}\right)$$

$$= 21 - 2 - 6$$

$$= 13$$

11. Official Ans. by NTA (1)

Sol. $(2e^{i\pi/6})^{100} = 2^{99}(p + iq)$

$$2^{100} \left(\cos \frac{50\pi}{3} + i \sin \frac{50\pi}{3} \right) = 2^{99}(p + iq)$$

$$p + iq = 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$p = -1, q = \sqrt{3}$$

$$x^2 - (\sqrt{3} - 1)x - \sqrt{3} = 0.$$

12. Official Ans. by NTA (6)

Sol. $\frac{(2i)^n}{(1-i)^{n-2}} = \frac{(2i)^n}{(-2i)^{\frac{n-2}{2}}}$

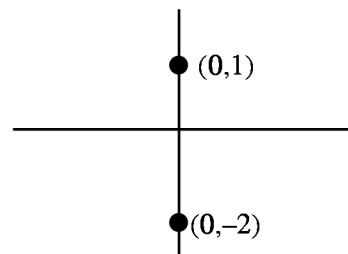
$$= \frac{(2i)^{\frac{n+2}{2}}}{(-1)^{\frac{n-2}{2}}} = \frac{2^{\frac{n+2}{2}} i^{\frac{n+2}{2}}}{(-1)^{\frac{n-2}{2}}}$$

This is positive integer for $n = 6$

13. Official Ans. by NTA (4)

Sol. Given $\frac{z-i}{z+2i} \in \mathbb{R}$

Then $\arg\left(\frac{z-i}{z+2i}\right)$ is 0 or π



$\Rightarrow S$ is straight line in complex

14. Official Ans. by NTA (6)

Sol. $|z - 3| = \operatorname{Re}(z)$

let $Z = x + iy$

$$\Rightarrow (x - 3)^2 + y^2 = x^2$$

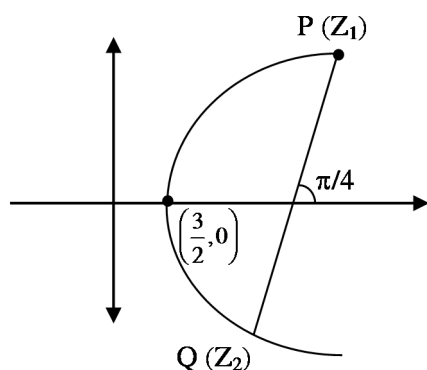
$$\Rightarrow x^2 + 9 - 6x + y^2 = x^2$$

$$\Rightarrow y^2 = 6x - 9$$

$$\Rightarrow y^2 = 6\left(x - \frac{3}{2}\right)$$

$\Rightarrow z_1$ and z_2 lie on the parabola mentioned in eq.(1)

$$\arg(z_1 - z_2) = \frac{\pi}{4}$$

$$\Rightarrow \text{Slope of PQ} = 1.$$


Let $P\left(\frac{3}{2} + \frac{3}{2}t_1^2, 3t_1\right)$ and $Q\left(\frac{3}{2} + \frac{3}{2}t_2^2, 3t_2\right)$

$$\text{Slope of PQ} = \frac{3(t_2 - t_1)}{\frac{3}{2}(t_1^2 - t_2^2)} = 1$$

$$\Rightarrow \frac{2}{t_2 + t_2} = 1$$

$$\Rightarrow t_2 + t_1 = 2$$

$$\text{Im}(z_1 + z_2) = 3t_1 + 3t_2 = 3(t_1 + t_2) = 3 \quad (2)$$

Ans. 6.00

Aliter :

Let $z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$

$$Z_1 - Z_2 = (X_1 - X_2) + i(Y_1 - Y_2)$$

$$\therefore \arg(z_1 - z_2) = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \frac{\pi}{4}$$

$$y_1 - y_2 = x_1 - x_2 \quad (1)$$

$$|z_1 - 3| = \text{Re}(z_1) \Rightarrow (x_1 - 3)^2 + y_1^2 = x_1^2 \quad (2)$$

$$|z_2 - 3| = \text{Re}(z_2) \Rightarrow (x_2 - 3)^2 + y_2^2 = x_2^2 \quad \text{---(2)}$$

sub (2) & (3)

$$(x_1 - 3)^2 - (x_2 - 3)^2 + y_1^2 - y_2^2 = x_1^2 - x_2^2$$

$$(x_1 - x_2)(x_1 + x_2 - 6) + (y_1 - y_2)(y_1 + y_2)$$

$$= (x_1 - x_2)(x_1 + x_2)$$

$$x_1 + x_2 - 6 + y_1 + y_2 = x_1 + x_2 \Rightarrow y_1 + y_2 = 6.$$

15. Official Ans. by NTA (98)

Sol. Let $z = x + iy$

$$\arg\left(\frac{x-2+iy}{x+2+iy}\right) = \frac{\pi}{4}$$

$$\arg (x-2+iy)-\arg (x+2+iy)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{y}{x-2}\right) - \tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{4}$$

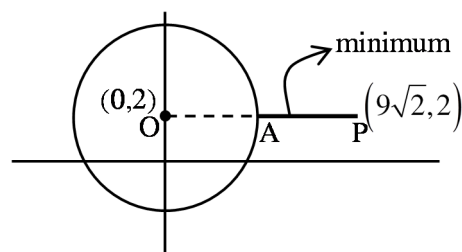
$$\frac{\frac{y}{x-2} - \frac{y}{x+2}}{1 + \left(\frac{y}{x-2}\right) \cdot \left(\frac{y}{x+2}\right)} = \tan \frac{\pi}{4} = 1$$

$$\frac{xy + 2y - xy + 2y}{x^2 - 4 + y^2} = 1$$

$$4y = x^2 - 4 + y^2$$

$$x^2 + y^2 - 4y - 4 = 0$$

locus is a circle with center $(0, 2)$ & radius $= 2\sqrt{2}$



$$\text{min. value} = (\text{AP})^2 = (\text{OP} - \text{OA})^2$$

$$= (9\sqrt{2} - 2\sqrt{2})^2$$

$$= (7\sqrt{2})^2 = 98$$

16. Official Ans. by NTA (4)

Sol. $\frac{z-i}{z-1}$ is purely Imaginary number

Let $z = x + iy$

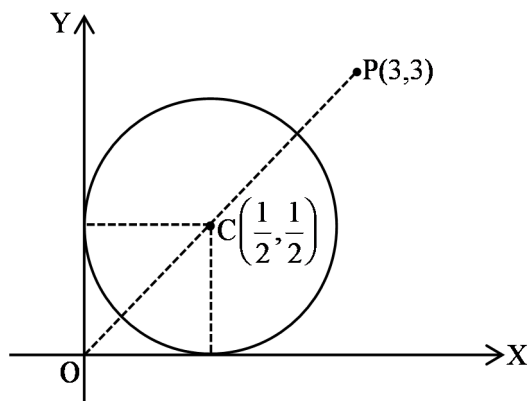
$$\therefore \frac{x + i(y-1)}{(x-1) + i(y)} \times \frac{(x-1) - iy}{(x-1) - iy}$$

$$\Rightarrow \frac{x(x-1) + y(y-1) + i(-y-x+1)}{(x-1)^2 + y^2} \text{ is purely}$$

Imaginary number

$$\Rightarrow x(x-1) + y(y-1) = 0$$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$



$$\therefore |z - (3 + 3i)|_{\min} = |PC| - \frac{1}{\sqrt{2}}$$

$$= \frac{5}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

17. Official Ans. by NTA (5)

Sol. $|z - 2 - 2i| \leq 1$

$$|x + iy - 2 - 2i| \leq 1$$

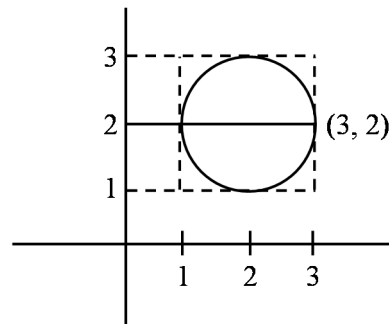
$$|(x-2) + i(y-2)| \leq 1$$

$$(x-2)^2 + (y-2)^2 \leq 1$$

$$|3iz + 6|_{\max} \text{ at } a + ib$$

$$|3i| \left| z + \frac{6}{3i} \right|$$

$$3|z - 2i|_{\max}$$



From Figure maximum distance at $3 + 2i$

$$a + ib = 3 + 2i = a + b = 3 + 2 = 5 \text{ Ans.}$$

18. Official Ans. by NTA (310)

$$\text{Sol. } K = \frac{1}{2^9} \left[\frac{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{21}}{\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{24}} + \frac{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{21}}{\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{24}} \right]$$

$$K = \frac{1}{512} \left[\frac{\left(e^{i\frac{2\pi}{3}}\right)^{21}}{\left(e^{-i\frac{\pi}{4}}\right)^{24}} + \frac{\left(e^{i\frac{\pi}{3}}\right)^{21}}{\left(e^{i\frac{\pi}{4}}\right)^{24}} \right]$$

$$K = \frac{1}{512} \left[e^{i(14\pi + 6\pi)} + e^{i(7\pi - 6\pi)} \right]$$

$$K = \frac{1}{512} \left[e^{20\pi i} + e^{\pi i} \right]$$

$$K = \frac{1}{512} [1 + (-1)] = 0$$

$$n = [k] = 0$$

$$\sum_{j=0}^5 (j+5)^2 - \sum_{j=0}^5 (j+5)$$

$$\sum_{j=0}^5 (j^2 + 25 + 10j - j - 5)$$

$$\sum_{j=0}^5 (j^2 + 9j + 20)$$

$$\sum_{j=0}^5 j^2 + 9 \sum_{j=0}^5 j + 20 \sum_{j=0}^5 1$$

$$\frac{5 \times 6 \times 11}{6} + 9 \left(\frac{5 \times 6}{2} \right) + 20 \times 6$$

$$= 55 + 135 + 120$$

$$= 310$$

19. Official Ans. by NTA (10)

Sol. Put $z = x + iy$

$$x + iy + \alpha|x + iy - 1| + 2i = 0$$

$$\Rightarrow x + \alpha\sqrt{(x-1)^2 + y^2} + i(y+2) = 0 + 0i$$

$$\Rightarrow y + 2 = 0 \text{ and } x + \alpha\sqrt{(x-1)^2 + y^2} = 0$$

$$\Rightarrow y = -2 \text{ and } \alpha^2 = \frac{x^2}{x^2 - 2x + 5}$$

$$\text{Now } \frac{x^2}{x^2 - 2x + 5} \in \left[0, \frac{5}{4}\right]$$

$$\therefore \alpha^2 \in \left[0, \frac{5}{4}\right] \Rightarrow \alpha \in \left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$$

$$\therefore p = -\frac{\sqrt{5}}{2}; q = \frac{\sqrt{5}}{2}$$

$$\Rightarrow 4(p^2 + q^2) = 4\left(\frac{5}{4} + \frac{5}{4}\right) = 10$$

20. Official Ans. by NTA (4)

Sol. (i) $(2 - i)z = (2 + i)\bar{z}$

$$\boxed{y = \frac{x}{2}}$$

$$(ii) (2 + i)z + (i - 2)\bar{z} - 4i = 0$$

$$\boxed{x + 2y = 2}$$

$$(iii) iz + \bar{z} + 1 + i = 0$$

$$\text{Eqn of tangent } \boxed{x - y + 1 = 0}$$

Solving (i) and (ii)

$$x = 1, y = \frac{1}{2}$$

$$\text{Now, } p = r \Rightarrow \left| \frac{1 - \frac{1}{2} + 1}{\sqrt{2}} \right| = r$$

$$\Rightarrow r = \frac{3}{2\sqrt{2}}$$

21. Official Ans. by NTA (48)

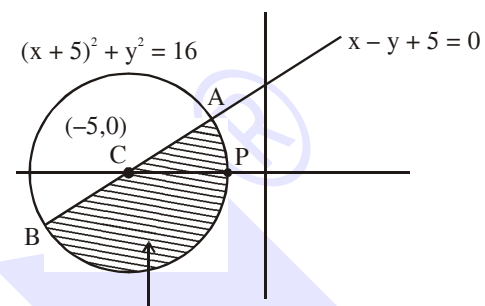
Sol. $|z + 5| \leq 4$

$$(x + 5)^2 + y^2 \leq 16 \quad \dots(1)$$

$$z(1+i) + \bar{z}(1-i) \geq -10$$

$$(z + \bar{z}) + i(z - \bar{z}) \geq -10$$

$$x - y + 5 \geq 0 \quad \dots(2)$$



Region bounded by inequalities (1) & (2)

$$|z + 1|^2 = |z - (-1)|^2$$

Let $P(-1, 0)$

$$\boxed{|z + 1|_{\text{Max.}}^2 = PB^2} \text{ (where B is in 3rd quadrant)}$$

for point of intersection

$$\begin{cases} (x+5)^2 + y^2 = 16 \\ x - y + 5 = 0 \end{cases} \Rightarrow y = \pm 2\sqrt{2}$$

$$A(2\sqrt{2} - 5, 2\sqrt{2}) \quad B(-2\sqrt{2} - 5, -2\sqrt{2})$$

$$PB^2 = (+2\sqrt{2} + 4)^2 + (2\sqrt{2})^2$$

$$|z + 1|^2 = 8 + 16 + 16\sqrt{2} + 8$$

$$\alpha + \beta\sqrt{2} = 32 + 16\sqrt{2}$$

$$\alpha = 32, \beta = 16 \Rightarrow \alpha + \beta = 48$$

22. Official Ans. by NTA (3)

Sol. $x^3 - 2x^2 + 2x - 1 = 0$
 $x = 1$ satisfying the equation
 $\therefore x - 1$ is factor of
 $x^3 - 2x^2 + 2x - 1$
 $= (x - 1)(x^2 - x + 1) = 0$
 $x = 1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}$
 $x = 1, -\omega^2, -\omega$
 sum of 162^{th} power of roots
 $= (1)^{162} + (-\omega^2)^{162} + (-\omega)^{162}$
 $= 1 + (\omega)^{324} + (\omega)^{162}$
 $= 1 + 1 + 1 = 3$
 $n + m = 45$

23. Official Ans by NTA (1)

Sol. $\exp \left(\frac{(|z|+3)(|z|-1)}{|z|+1} \ln 2 \right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|$
 $\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{|z|+1}} \geq \log_{\sqrt{2}} (16)$
 $\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{|z|+1}} \geq 2^3$
 $\Rightarrow \frac{(|z|+3)(|z|-1)}{|z|+1} \geq 3$
 $\Rightarrow (|z|+3)(|z|-1) \geq 3(|z|+1)$
 $|z|^2 + 2|z| - 3 \geq 3|z| + 3$
 $\Rightarrow |z|^2 + |z| - 6 \geq 0$
 $\Rightarrow (|z|-3)(|z|+2) \geq 0 \Rightarrow |z|-3 \geq 0$
 $\Rightarrow |z| \geq 3 \Rightarrow |z|_{\min} = 3$

24. Official Ans. by NTA (2)

Sol. $\log_{\frac{1}{\sqrt{2}}} \left(\frac{|z|+11}{(|z|-1)^2} \right) \leq 2$
 $\frac{|z|+11}{(|z|-1)^2} \geq \frac{1}{2}$
 $2|z| + 22 \geq (|z|-1)^2$
 $2|z| + 22 \geq |z|^2 + 1 - 2|z|$
 $|z|^2 - 4|z| - 21 \leq 0$
 $\Rightarrow |z| \leq 7$
 $\therefore$ Largest value of $|z|$ is 7

25. Official Ans. by NTA (36)

Sol. Let $M = (P^{-1}AP - I)^2$
 $= (P^{-1}AP)^2 - 2P^{-1}AP + I$
 $= P^{-1}A^2P - 2P^{-1}AP + I$
 $PM = A^2P - 2AP + P$
 $= (A^2 - 2A.I + I^2)P$
 $\Rightarrow \text{Det}(PM) = \text{Det}((A - I)^2 \times P)$
 $\Rightarrow \text{Det}P \cdot \text{Det}M = \text{Det}(A - I)^2 \times \text{Det}(P)$
 $\Rightarrow \text{Det}M = (\text{Det}(A - I))^2$

$$\text{Now } A - I = \begin{bmatrix} 1 & 7 & w^2 \\ -1 & -w-1 & 1 \\ 0 & -w & -w \end{bmatrix}$$

$\text{Det}(A - I) = (w^2 + w + w) + 7(-w) + w^3 = -6w$
 $\text{Det}((A - I)^2) = 36w^2$
 $\Rightarrow \alpha = 36$

26. Official Ans. by NTA (4)

Sol. $\omega = z\bar{z} - 2z + 2$

$$\left| \frac{z+i}{z-3i} \right| = 1$$

$$\Rightarrow |z+i| = |z-3i|$$

$$\Rightarrow z = x + i, x \in \mathbb{R}$$

$$\omega = (x+i)(x-i) - 2(x+i) + 2$$

$$= x^2 + 1 - 2x - 2i + 2$$

$$\text{Re}(\omega) = x^2 - 2x + 3$$

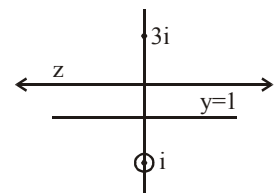
For min $(\text{Re}(\omega))$, $x = 1$

$$\Rightarrow \omega = 2 - 2i = 2(1 - i) = 2\sqrt{2} e^{-i\frac{\pi}{4}}$$

$$\omega^n = (2\sqrt{2})^n e^{-i\frac{n\pi}{4}}$$

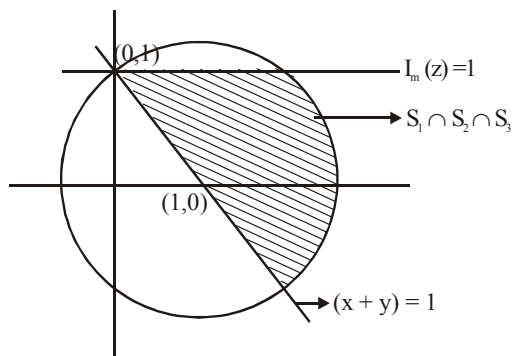
For real & minimum value of n ,

$$n = 4$$



27. Official Ans. by NTA (3)

Sol. For $|z-1| \leq \sqrt{2}$, z lies on and inside the circle of radius $\sqrt{2}$ units and centre $(1, 0)$.



For S_2

Let $z = x + iy$

Now, $(1-i)(z) = (1-i)(x+iy)$

$\text{Re}((1-i)z) = x+y$

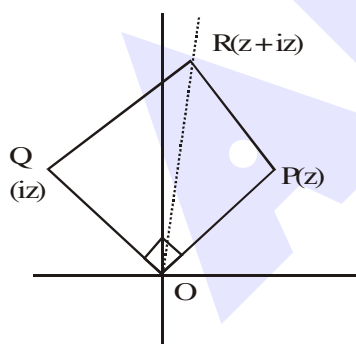
$\Rightarrow x+y \geq 1$

$\Rightarrow S_1 \cap S_2 \cap S_3$ has infinity many elements

Ans. (3)

28. Official Ans. by NTA (2)

Sol.



$$A = \frac{1}{2} |z| |iz|$$

$$= \frac{|z|^2}{2}$$

29. Official Ans. by NTA (2)

Sol. $az\bar{z} + \alpha\bar{z} + \bar{\alpha}z + d = 0 \rightarrow \text{Circle}$

$$\text{centre} = \frac{-\alpha}{a} \quad 2 = \sqrt{\frac{\alpha\bar{\alpha}}{a^2} - \frac{d}{a}} = \sqrt{\frac{\alpha\bar{\alpha} - ad}{a^2}}$$

So $|\alpha|^2 - ad > 0$ & $a \in \mathbb{R} - \{0\}$

30. Official Ans. by NTA (6)

Sol. If $0, z_1, z_2$ are vertices of equilateral triangles

$$\Rightarrow a^2 + z_1^2 + z_2^2 = 0 \quad (z_1 + z_2) + z_1 z_2$$

$$\Rightarrow (z_1 + z_2)^2 = 3z_1 z_2$$

$$\Rightarrow a^2 = 3 \times 12$$

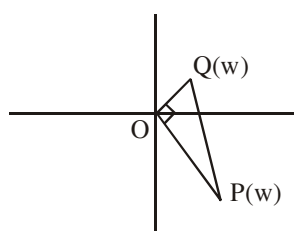
$$\Rightarrow |a| = 6$$

31. Official Ans. by NTA (2)

Sol. $w = 1 - \sqrt{3}i \Rightarrow |w| = 2$

$$\text{Now, } |z| = \frac{1}{|w|} \Rightarrow |z| = \frac{1}{2}$$

$$\text{and } \text{amp}(z) = \frac{\pi}{2} + \text{amp}(w)$$



$$\Rightarrow \text{Area of triangle} = \frac{1}{2} \cdot \text{OP} \cdot \text{OQ}$$

$$= \frac{1}{2} \cdot 2 \cdot \frac{1}{2} = \frac{1}{2}$$

32. Official Ans. by NTA (0)

Sol. $P(x) = f(x^3) + xg(x^3)$

$$P(1) = f(1) + g(1) \quad \dots(1)$$

Now $P(x)$ is divisible by $x^2 + x + 1$

$$\Rightarrow P(x) = Q(x)(x^2 + x + 1)$$

$P(w) = 0 = P(w^2)$ where w, w^2 are non-real cube roots of unity

$$P(x) = f(x^3) + xg(x^3)$$

$$P(w) = f(w^3) + wg(w^3) = 0$$

$$f(1) + wg(1) = 2 \quad \dots(2)$$

$$P(w^2) = f(w^6) + w^2g(w^6) = 0$$

$$f(1) + w^2g(1) = 0 \quad \dots(3)$$

$$(2) + (3)$$

$$\Rightarrow 2f(1) + (w + w^2)g(1) = 0$$

$$2f(1) = g(1) \quad \dots(4)$$

$$(2) - (3)$$

$$\Rightarrow (w - w^2)g(1) = 0$$

$$g(1) = 0 = f(1) \quad \text{from (4)}$$

$$\text{from (1)} P(1) = f(1) + g(1) = 0$$