

SOLUTION OF TRIANGLE

Paragraph for Question No. 1 and 2

Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

1. Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is [JEE(Advanced) 2023]
2. Then the inradius of the triangle ABC is [JEE(Advanced) 2023]
3. Let PQRS be a quadrilateral in a plane, where $QR = 1$, $\angle PQR = \angle QRS = 70^\circ$, $\angle PQS = 15^\circ$ and $\angle PRS = 40^\circ$. If $\angle RPS = \theta^\circ$, $PQ = \alpha$ and $PS = \beta$, then the interval(s) that contain(s) the value of $4\alpha\beta \sin\theta^\circ$ is/are [JEE(Advanced) 2022]

- (A) $(0, \sqrt{2})$ (B) (1, 2)
 (C) $(\sqrt{2}, 3)$ (D) $(2\sqrt{2}, 3\sqrt{2})$

4. In a triangle ABC, let $AB = \sqrt{23}$, $BC = 3$ and $CA = 4$. Then the value of $\frac{\cot A + \cot C}{\cot B}$ is _____. [JEE(Advanced) 2021]
5. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) TRUE? [JEE(Advanced) 2021]

- (A) $\cos P \geq 1 - \frac{p^2}{2qr}$ (B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$
 (C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$ (D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

6. Let x, y and z be positive real numbers. Suppose x, y and z are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE? [JEE(Advanced) 2020]

- (A) $2Y = X + Z$ (B) $Y = X + Z$
 (C) $\tan \frac{X}{2} = \frac{x}{y+z}$ (D) $x^2 + z^2 - y^2 = xz$

7. In a non-right-angled triangle ΔPQR , let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct? [JEE(Advanced) 2019]

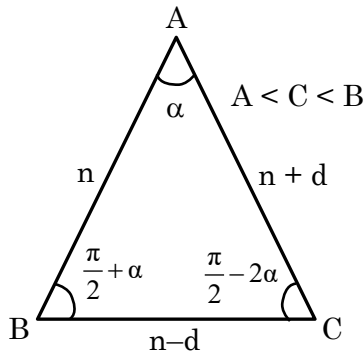
- (A) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$ (B) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2 - \sqrt{3})$
 (C) Length of RS = $\frac{\sqrt{7}}{2}$ (D) Length of OE = $\frac{1}{6}$

8. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE? [JEE(Advanced) 2018]
- (A) $\angle QPR = 45^\circ$
- (B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$
- (C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$
- (D) The area of the circumcircle of the triangle PQR is 100π .
9. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then- [JEE(Advanced) 2017]
- (A) area of the triangle XYZ is $6\sqrt{6}$
- (B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
- (C) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
- (D) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$
10. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is a third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is - [JEE(Advanced) 2014]
- (A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

SOLUTIONS

1. Ans. (1008.00)

Sol.



$$n - d = 2 \sin \alpha \quad \dots(1)$$

$$n + d = 2 \sin \left(\frac{\pi}{2} + \alpha \right)$$

$$\Rightarrow n + d = 2 \cos \alpha \quad \dots(2)$$

$$n = 2 \sin \left(\frac{\pi}{2} - 2\alpha \right)$$

$$\Rightarrow n = 2 \cos 2\alpha \quad \dots(3)$$

$$\Rightarrow 2 \cos 2\alpha = \sin \alpha + \cos \alpha$$

$$\Rightarrow 2(\cos \alpha - \sin \alpha) = 1$$

$$\Rightarrow \sin 2\alpha = \frac{3}{4}$$

Then,

$$a = \frac{1}{2} \cdot n \cdot (n + d) \cdot \sin \alpha = \frac{1}{2} \cdot 2 \cos 2\alpha \cdot 2 \cos \alpha \cdot \sin \alpha$$

$$= \sin 2\alpha \cdot \cos 2\alpha$$

$$= \frac{3}{4} \times \frac{\sqrt{7}}{4} = \frac{3\sqrt{7}}{16}$$

$$(64a)^2 = \left(64 \times \frac{3\sqrt{7}}{16} \right)^2 = 16 \times 9 \times 7 = 1008$$

2. Ans. (0.25)

Sol. From above equation in Ques. 1

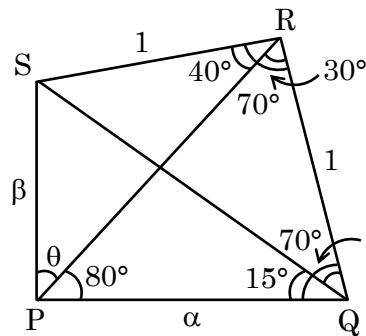
$$r = \frac{\Delta}{s} = \frac{1}{2} \frac{n(n+d) \sin \alpha}{\left(\frac{3n}{2} \right)}$$

$$= \frac{(n+d) \cdot \sin \alpha}{3} = \frac{2 \cos \alpha \cdot \sin \alpha}{3} \quad (\text{from (2)})$$

$$r = \frac{\sin 2\alpha}{3} = \frac{1}{4}$$

3. Ans. (A, B)

Sol.



$$\angle PRQ = 70^\circ - 40^\circ = 30^\circ$$

$$\angle RQS = 70^\circ - 15^\circ = 55^\circ$$

$$\angle QSR = 180^\circ - 55^\circ - 70^\circ = 55^\circ$$

$$\therefore QR = RS = 1$$

$$\angle QPR = 180^\circ - 70^\circ - 30^\circ = 80^\circ$$

Apply sine-rule in \$\Delta PRQ\$:

$$\frac{\alpha}{\sin 30^\circ} = \frac{1}{\sin 80^\circ} \Rightarrow \alpha = \frac{1}{2 \sin 80^\circ} \quad \dots(1)$$

Apply sine-rule in \$\Delta PRS\$

$$\frac{\beta}{\sin 40^\circ} = \frac{1}{\sin \theta} \Rightarrow \beta \sin \theta = \sin 40^\circ \quad \dots(2)$$

$$4\alpha\beta \sin \theta = \frac{4 \sin 40^\circ}{2 \sin 80^\circ} = \frac{4 \sin 40^\circ}{2(2 \sin 40^\circ \cos 40^\circ)}$$

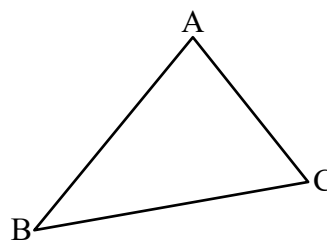
$$= \sec 40^\circ$$

$$\text{Now } \sec 30^\circ < \sec 40^\circ < \sec 45^\circ$$

$$\Rightarrow \frac{2}{\sqrt{3}} < \sec 40^\circ < \sqrt{2}$$

4. Ans. (2)

Sol.



$$\text{Given } c = \sqrt{23}; a = 3; b = 4$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2 + c^2 - a^2}{2bc \sin A}$$

$$= \frac{b^2 + c^2 - a^2}{2.2\Delta} \left\{ \Delta = \frac{1}{2} bc \sin A \right\}$$

$$\cot A = \frac{b^2 + c^2 - a^2}{4\Delta}$$

Similarly, $\cot B = \frac{a^2 + c^2 - b^2}{4\Delta}$

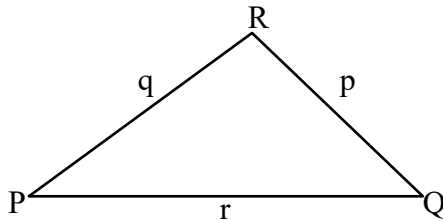
& $\cot C = \frac{a^2 + b^2 - c^2}{4\Delta}$

$$\therefore \frac{\cot A + \cot C}{\cot B} = \frac{b^2 + c^2 - a^2 + a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$$

$$= \frac{2b^2}{a^2 + c^2 - b^2} = \frac{32}{16} = 2$$

5. Ans. (A, B)

Sol.



(A)

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr} = \frac{q^2 + r^2}{2qr} - \frac{p^2}{2qr} \geq 1 - \frac{p^2}{2qr}$$

(as $p^2 + q^2 \geq 2qr$ (AM \geq GM)),

So (A) is correct

(B) $(p + q) \cos R \geq (q - r) \cos P + (p - r) \cos Q$

$$\Rightarrow (p \cos R + r \cos P) + (q \cos R + r \cos Q) \geq q \cos P + p \cos Q$$

$$\Rightarrow q + p \geq r$$

So (B) is correct

(C) $\frac{q+r}{p} = \frac{\sin Q + \sin R}{\sin P} \geq \frac{2\sqrt{\sin Q \times \sin R}}{\sin P}$

So (C) is incorrect

(D) $\cos Q > \frac{p}{r} \Rightarrow \sin R \cos Q > \sin P$

$$\Rightarrow \sin P + \sin(R - Q) > 2 \sin P$$

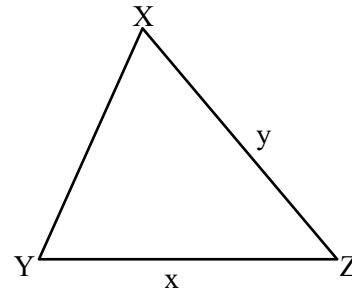
$$\Rightarrow \sin(R - Q) > \sin P$$

need not necessarily hold true if $R < Q$

Hence (A), (B)

6. Ans. (B, C)

Sol.



$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x+y+z}$$

$$\frac{\Delta}{S(S-x)} + \frac{\Delta}{S(S-z)} = \frac{2y}{2S}$$

$$\frac{\Delta}{S} \left(\frac{2S - (x+z)}{(S-x)(S-z)} \right) = \frac{y}{S}$$

$$\Rightarrow \frac{\Delta y}{S(S-x)(S-z)} = \frac{y}{S}$$

$$\Rightarrow \Delta^2 = (S-x)^2 (S-z)^2$$

$$\Rightarrow S(S-y) = (S-x)(S-z)$$

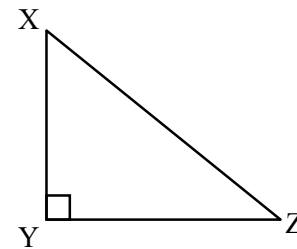
$$\Rightarrow (x+y+z)(x+z-y) = (y+z-x)(x+y-z)$$

$$\Rightarrow (x+z)^2 - y^2 = y^2 - (z-x)^2$$

$$\Rightarrow (x+z)^2 + (x-z)^2 = 2y^2$$

$$\Rightarrow x^2 + z^2 = y^2 \Rightarrow \angle Y = \frac{\pi}{2}$$

$$\Rightarrow \angle Y = \angle X + \angle Z$$



$$\tan \frac{X}{2} = \frac{\Delta}{S(S-x)}$$

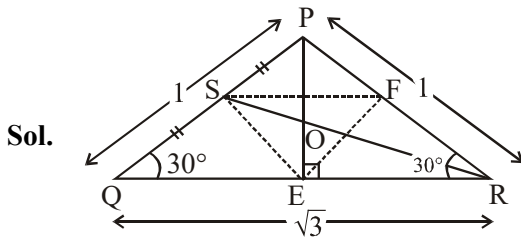
$$\tan \frac{X}{2} = \frac{\frac{1}{2}xz}{\frac{(y+z)^2 - x^2}{4}}$$

$$\tan \frac{X}{2} = \frac{2xz}{y^2 + z^2 + 2yz - x^2}$$

$$\tan \frac{X}{2} = \frac{2xz}{2z^2 + 2yz} \quad (\text{using } y^2 = x^2 + z^2)$$

$$\tan \frac{X}{2} = \frac{x}{y+z}$$

7. Ans. (B, C, D)



Sol.

$$\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}$$

$$\Rightarrow P = \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ and } Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

Since $p > q \Rightarrow P > Q$

$$\text{So, if } P = \frac{\pi}{3} \text{ and } Q = \frac{\pi}{6} \Rightarrow R = \frac{\pi}{2}$$

(not possible)

$$\text{Hence, } P = \frac{2\pi}{3} \text{ and } Q = R = \frac{\pi}{6}$$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}(1)(\sqrt{3})\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}+2}{2}\right)} = \frac{\sqrt{3}}{2}(2-\sqrt{3})$$

$$\text{Now, area of } \Delta SEF = \frac{1}{4} \text{ area of } \Delta PQR$$

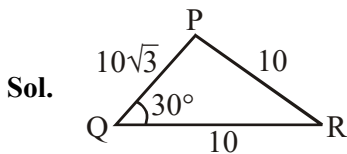
$$\Rightarrow \text{area of } \Delta SOE = \frac{1}{3} \text{ area of } \Delta SEF = \frac{1}{12}$$

$$\text{area of } \Delta PQR = \frac{1}{12} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$$

$$RS = \frac{1}{2} \sqrt{2(3)+2(1)-1} = \frac{\sqrt{7}}{2}$$

$$OE = \frac{1}{3} PE = \frac{1}{3} \cdot \frac{1}{2} \sqrt{2(1)^2+2(1)^2-3} = \frac{1}{6}$$

8. Ans. (B, C, D)



Sol.

$$\cos 30^\circ = \frac{(10\sqrt{3})^2 + (10)^2 - (PR)^2}{2 \times 10\sqrt{3} \times 10}$$

$$\Rightarrow PR = 10$$

$$\therefore QR = PR \Rightarrow \angle PQR = \angle QPR$$

$$\angle QPR = 30^\circ$$

(B) area of ΔPQR

$$= \frac{1}{2} \times 10\sqrt{3} \times 10 \times \sin 30^\circ = \frac{1}{2} \times 10 \times 10\sqrt{3} \times \frac{1}{2}$$

$$= 25\sqrt{3}$$

$$\angle QRP = 180^\circ - (30^\circ + 30^\circ) = 120^\circ$$

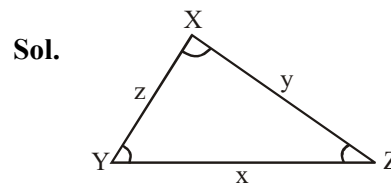
$$(C) r = \frac{\Delta}{s} = \frac{25\sqrt{3}}{\left(\frac{10+10+10\sqrt{3}}{2}\right)} = \frac{25\sqrt{3}}{10+5\sqrt{3}}$$

$$= 5\sqrt{3} \cdot (2-\sqrt{3}) = 10\sqrt{3} - 15$$

$$(D) R = \frac{a}{2\sin A} = \frac{10}{2\sin 30^\circ} = 10$$

$$\therefore \text{Area} = \pi R^2 = 100\pi$$

9. Ans. (A, C, D)



Sol.

$$\text{Let } \frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = k$$

$$s-x=4k \Rightarrow y+z-x=8k$$

$$s-y=3k \Rightarrow x+z-y=6k$$

$$s-z=2k \Rightarrow x+y-z=4k$$

$$\Rightarrow x=5k, y=6k, z=7k$$

$$\Rightarrow 3s-(x+y+z)=9k$$

$$s=9k$$

Let r be inradius

$$\Rightarrow \pi r^2 = \frac{8\pi}{3}$$

$$\pi \left(\frac{\Delta}{s}\right)^2 = \frac{8\pi}{3}$$

$$\Delta = \sqrt{\frac{8}{3}} \cdot s$$

$$\sqrt{s(s-x)(s-y)(s-z)} = \sqrt{\frac{8}{3}} \cdot s$$

$$\sqrt{9k \cdot 4k \cdot 3k \cdot 2k} = \sqrt{\frac{8}{3}} \cdot 9k$$

$$\sqrt{24 \cdot 9k^2} = \sqrt{\frac{8}{3}} \cdot 9k$$

$$k=1$$

$$\Rightarrow x = 5, y = 6, z = 7$$

$$\Delta = \sqrt{\frac{8}{3}} \cdot 9k = \sqrt{\frac{8}{3}} \cdot 9 = 6\sqrt{6}$$

$$R = \text{circumradius} = \frac{xyz}{4\Delta} = \frac{5 \cdot 6 \cdot 7}{4 \cdot 6\sqrt{6}} = \frac{35}{4\sqrt{6}}$$

$$\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{r}{4R} = \frac{\frac{\Delta}{s}}{\frac{xyz}{\Delta}} = \frac{\Delta^2}{s \cdot xyz}$$

$$= \frac{36.6}{9 \cdot 5 \cdot 6 \cdot 7} = \frac{4}{35}$$

$$\cos Z = \frac{x^2 + y^2 - z^2}{2xy} = \frac{25 + 36 - 49}{2 \cdot 5 \cdot 6} = \frac{1}{5}$$

$$\sin^2 \left(\frac{X+Y}{2} \right) = \cos^2 \frac{Z}{2} = \frac{1 + \cos Z}{2} = \frac{1 + \frac{1}{5}}{2} = \frac{3}{5}$$

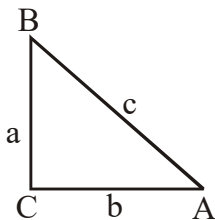
10. Ans. (B)

Sol. $a + b = x$

$$ab = y$$

$$(a + b)^2 - c^2 = ab$$

$$\Rightarrow a^2 + b^2 - c^2 = -ab$$



$$\cos c = -\frac{1}{2} \Rightarrow c = \frac{2\pi}{3}$$

$$\text{Now, } \frac{r}{R} = \frac{\frac{\Delta}{s}}{\frac{abc}{4\Delta}} = \frac{4\Delta^2}{(a+b+c)abc}$$

$$= \frac{8 \left(\frac{1}{2} ab \frac{\sqrt{3}}{2} \right)^2}{(x+c)(yc)} = \frac{\frac{1}{2} y^2 \cdot 3}{(x+c)cy} = \frac{3y}{2c(x+c)}$$