MATRIX

1. Let $M = (a_{ij})$, $i, j \in \{1, 2, 3\}$, be the 3 × 3 matrix such that $a_{ij} = 1$ if $j+1$ is divisible by i, otherwise $a_{ij} = 0$. Then which of the following statements is (are) true ? **[JEE(Advanced) 2023]**

(A) M is invertible

(B) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $M\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$

(C) The set $\{X \in \mathbb{R}^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(D) The matrix (M – 2I) is invertible, where I is the 3 × 3 identity matrix

2. Let $R = \left\{ \begin{pmatrix} a & 3 & b \\ c & 2 & d \\ 0 & 5 & 0 \end{pmatrix} : a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\} \right\}$. Then the number of invertible matrices in R is

[JEE(Advanced) 2023]

3. Let β be a real number. Consider the matrix

$$A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$$

If $A^7 - (\beta - 1)A^6 - \beta A^5$ is a singular matrix, then the value of 9β is ______. **[JEE(Advanced) 2022]**

4. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then which of the following matrices is equal to M^{2022} ? **[JEE(Advanced) 2022]**

(A) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$

(B) $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$

(C) $\begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$

(D) $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

5. For any 3 × 3 matrix M, let |M| denote the determinant of M. Let **[JEE(Advanced) 2021]**

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3 × 3, then which of the following statements is (are) **TRUE** ?

(A) F = PEP and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $\left|(EF)^3\right| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

6. For any 3 × 3 matrix M, let |M| denote the determinant of M. Let I be the 3 × 3 identity matrix. Let E and F be two 3 × 3 matrices such that (I – EF) is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE ?** **[JEE(Advanced) 2021]**

(A) $|FE| = |I - FE||FGE|$
(B) $(I - FE)(I + FGE) = I$
(C) EFG = GEF
(D) $(I - FE)(I - FGE) = I$

7. Let M be a 3 × 3 invertible matrix with real entries and let I denote the 3 × 3 identity matrix.
If M^{-1} = adj (adj M), then which of the following statement is/are ALWAYS TRUE ?

[JEE(Advanced) 2020]

(A) M = I
(B) det M = 1
(C) $M^2 = I$
(D) $(\text{adj } M)^2 = I$

8. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2 × 2 matrix such that the trace of A is 3 and the trace of A^3 is –18, then the value of the determinant of A is _____

[JEE(Advanced) 2020]

9. Let $M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$,

where α = α(θ) and β = β(θ) are real number, and I is the 2 × 2 identity matrix. If
α* is the minimum of the set {α(θ) : θ ∈ [0, 2π)} and
β* is the minimum of the set {β(θ) : θ ∈ [0, 2π)},
then the value of α* + β* is **[JEE(Advanced) 2019]**

(A) $-\frac{37}{16}$
(B) $-\frac{29}{16}$
(C) $-\frac{31}{16}$
(D) $-\frac{17}{16}$

10. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\text{adj}M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) a + b = 3
(B) $\det(\text{adj}M^2) = 81$
(C) $(\text{adj}M)^{-1} + \text{adj}M^{-1} = -M$
(D) If $M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then α – β + γ = 3

11. Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X = \sum_{k=1}^{6} P_K \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_K^T$

where P_K^T denotes the transpose of the matrix P_K. Then which of the following options is/are correct?

[JEE(Advanced) 2019]

(A) X – 30I is an invertible matrix
(B) The sum of diagonal entries of X is 18
(C) If $X\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then α = 30x
(D) X is a symmetric matrix

12. Let $x \in \mathbb{R}$ and let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$ and $R = PQP^{-1}$.

Then which of the following options is/are correct? **[JEE(Advanced) 2019]**

(A) For x = 1, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(B) There exists a real number x such that PQ = QP

(C) $\det R = \det\begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$, for all $x \in \mathbb{R}$

(D) For x = 0, if $R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, then a + b = 5

13. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced) 2018]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

14. Let P be a matrix of order 3 × 3 such that all the entries in P are from the set {–1, 0, 1}. Then, the maximum possible value of the determinant of P is _____ . **[JEE(Advanced) 2018]**

15. Which of the following is(are) NOT the square of a 3 × 3 matrix with real entries ? **[JEE(Advanced) 2017]**

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

16. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ **[JEE(Advanced) 2017]**

17. How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of M^TM is 5 ? **[JEE(Advanced) 2017]**

(A) 198 (B) 126

(C) 135 (D) 162

18. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$, Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in \mathbb{R}$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then- **[JEE(Advanced) 2016]**

(A) $\alpha = 0, k = 8$ (B) $4\alpha - k + 8 = 0$

(C) $\det(P\operatorname{adj}(Q)) = 2^9$ (D) $\det(Q\operatorname{adj}(P)) = 2^{13}$

19. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **[JEE(Advanced) 2016]**

(A) 52 (B) 103 (C) 201 (D) 205

20. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ? **[JEE(Advanced) 2015]**

(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$

(C) $X^4Z^3 - Z^3X^4$ (D) $X^{23} + Y^{23}$

21. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if **[JEE(Advanced) 2014]**

(A) the first column of M is the transpose of the second row of M

(B) the second row of M is the transpose of the first column of M

(C) M is a diagonal matrix with nonzero entries in the main diagonal

(D) the product of entries in the main diagonal of M is not the square of an integer

22. Let M and N be two 3×3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then **[JEE(Advanced) 2014]**

(A) determinant of $(M^2 + MN^2)$ is 0

(B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is zero matrix

(C) determinant of $(M^2 + MN^2) \geq 1$

(D) for a 3×3 matrix U, if $(M^2 + MN^2)$ U equals the zero matrix then U is the zero matrix

SOLUTIONS

1. Ans. (B, C)

Sol. $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$|M| = -1 + 1 = 0$

$\Rightarrow$ M is singular so non-invertible

Option (B) :

$$M\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix}$$

$$\left.\begin{matrix} a_1 + a_2 + a_3 = -a_1 \\ a_1 + a_3 = -a_2 \\ a_2 = -a_3 \end{matrix}\right\} \Rightarrow a_1 = 0 \text{ and } a_2 + a_3 = 0$$

infinite solutions exists [B] is correct.

Option (D) :

$$M - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$|M - 2I| = 0 \Rightarrow$ [D] is wrong

Option (C) :

$$MX = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0$

$x + z = 0$

$y = 0$

$\therefore$ Infinite solution

[C] is correct

2. Ans. (3780)

Sol. Let us calculate when $|R| = 0$

Case-I $ad = bc = 0$

Now $ad = 0$

$\Rightarrow$ Total – (When none of a & d is 0)

$= 8^2 - 7^2 = 15$ ways

Similarly $bc = 0 \Rightarrow 15$ ways

$\therefore 15 \times 15 = 225$ ways of $ad = bc = 0$

Case-II $ad = bc \neq 0$

either $a = d = b = c$

OR $a \neq d$, $b \neq d$ but $ad = bc$

${}^7C_1 = 7$ ways

${}^7C_2 \times 2 \times 2 = 84$ ways

Total 91 ways

$\therefore |R| = 0$ in $225 + 91 = 316$ ways

$|R| \neq 0$ in $8^4 - 316 = 3780$

3. Ans. (3)

Sol. $A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$ $|A| = -1$

$\Rightarrow |A^7 - (\beta - 1)A^6 - \beta A^5| = 0$

$\Rightarrow |A|^5 |A^2 - (\beta - 1)A - \beta I| = 0$

$\Rightarrow |A|^5 |(A^2 - \beta A) + A - \beta I| = 0$

$\Rightarrow |A|^5 |A(A - \beta I) + I(A - \beta I)| = 0$

$|A|^5 |(A + I)(A - \beta I)| = 0$

$$A + I = \begin{pmatrix} \beta+1 & 0 & 1 \\ 2 & 2 & -2 \\ 3 & 1 & -1 \end{pmatrix} \Rightarrow |A + I| = -4, \text{ Here}$$

$|A| \neq 0$ & $|A + I| \neq 0$

$$A - \beta I = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1-\beta & -2 \\ 3 & 1 & -2-\beta \end{pmatrix}$$

$$|A - \beta I| = 2 - 3(1 - \beta) = 3\beta - 1 = 0 \Rightarrow \beta = \frac{1}{3}$$

$9\beta = 3$

4. Ans. (A)

Sol. $M = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$

$$M = \begin{bmatrix} \frac{3}{2}+1 & \frac{3}{2} \\ \frac{-3}{2} & \frac{-3}{2}+1 \end{bmatrix}$$

$$M = I + \frac{3}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M^{2022} = \left(I + \frac{3}{2}A\right)^{2022}$$

$= I + 3033A$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3033\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3034 & 3033 \\ -3033 & -3032 \end{bmatrix}$$

5. Ans.(A, B, D)

Sol. $$PEP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) $\left|EQ + PFQ^{-1}\right| = \left|EQ\right| + \left|PFQ^{-1}\right|$

$|E| = 0$ and $|F| = 0$ and $|Q| \neq 0$

$|EQ| = |E||Q| = 0$, $\left|PFQ^{-1}\right| = \dfrac{|P||F|}{|Q|} = 0$

$T = EQ + PFQ^{-1}$

$TQ = EQ^2 + PF = EQ^2 + P^2EP = EQ^2 + EP$

$= E(Q^2 + P)$

$|TQ| = \left|E\left(Q^2 + P\right)\right| \Rightarrow |T||Q|$

$= |E|\left|Q^2 + P\right| = 0 \Rightarrow |T| = 0$ (as $|Q| \neq 0$)

(C) $\left|(EF)^3\right| > |EF|^2$

Here $0 > 0$ (false)

(D) as $P^2 = I \Rightarrow P^{-1} = P$ so $P^{-1}FP = PFP$

$= PPEPP = E$

so $E + P^{-1}FP = E + E = 2E$

$P^{-1}EP + F \Rightarrow PEP + F = 2PEP$

$Tr(2PEP) = 2Tr(PEP) = 2Tr(EPP) = 2Tr(E)$

6. Ans. (A, B, C)

Sol. $|I - EF| \neq 0$; $G = (I - EF)^{-1} \Rightarrow G^{-1} = I - EF$

Now, $G.G^{-1} = I = G^{-1}G$

$\Rightarrow G(I - EF) = I = (I - EF)G$

$\Rightarrow G - GEF = I = G - EFG$

$\Rightarrow GEF = EFG$ [C is Correct]

$(I - FE)(I + FGE) = I + FGE - FE - FEFGE$

$= I + FGE - FE - F(G - I)E$

$= I + FGE - FE - FGE + FE$

$= I$ [(B) is Correct]

(So 'D' is Incorrect)

We have

$(I - FE)(I + FGE) = I$ …..(I)

Now

$FE(I + FGE)$

$= FE + FEFGE$

$= FE + F(G - I)E$

$= FE + FGE - FE$

$= FGE$

$\Rightarrow |FE|\,|I + FGE| = |FGE|$

$\Rightarrow |FE| \times \dfrac{1}{|I - FE|} = |FGE|$ (from (1))

$\Rightarrow |FE| = |I - FE|\,|FGE|$

(option (A) is Correct)

7. Ans. (B, C, D)

Sol. $\det(M) \neq 0$

$M^{-1} = adj(adj\ M)$

$M^{-1} = \det(M).M$

$M^{-1}M = \det(M).M^2$

$I = \det(M).M^2$ …. (i)

$\det(I) = (\det(M))^5$

$1 = \det(M)$ …. (ii)

From (i) $I = M^2$

$(adj\ M)^2 = adj(M^2) = adj\ I = I$

8. Ans. (5)

Sol. M-I

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}$

$$A^3 = \begin{bmatrix} a^3+2abc+bdc & a^2b+abd+b^2c+bd^2 \\ a^2c+adc+bc^2+d^2c & abc+2bcd+d^3 \end{bmatrix}$$

Given trace(A) = a + d = 3

and trace(A^3) = $a^3 + d^3 + 3abc + 3bcd = -18$

$\Rightarrow$ $a^3 + d^3 + 3bc(a + d) = -18$

$\Rightarrow$ $a^3 + d^3 + 9bc = -18$

$\Rightarrow$ $(a + d)((a + d)^2 - 3ad) + 9bc = -18$

$\Rightarrow$ $3(9 - 3ad) + 9bc = -18$

$\Rightarrow$ ad – bc = 5 = determinant of A

M-II

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $\Delta = ad - bc$

$|A - \lambda I| = (a - \lambda)(d - \lambda) - bc$

$= \lambda^2 - (a + d)\lambda + ad - bc$

$= \lambda^2 - 3\lambda + \Delta$

$\Rightarrow$ $O = A^2 - 3A + \Delta I$

$\Rightarrow$ $A^2 = 3A - \Delta I$

$\Rightarrow$ $A^3 = 3A^2 - \Delta A$

$= 3(3A - \Delta I) - \Delta A$

$= (9 - \Delta)A - 3\Delta I$

$= (9 - \Delta)\begin{bmatrix} a & b \\ c & d \end{bmatrix} - 3\Delta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ trace $A^3 = (9 - \Delta)(a + d) - 6\Delta$

$\Rightarrow$ $-18 = (9 - \Delta)(3) - 6\Delta$

$= 27 - 9\Delta$

$\Rightarrow$ $9\Delta = 45 \Rightarrow \Delta = 5$

9. Ans. (B)

Sol. Given $M = \alpha I + \beta M^{-1}$

$\Rightarrow M^2 - \alpha M - \beta I = O$

By putting values of M and M^2, we get

$\alpha(\theta) = 1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{\sin^2 2\theta}{2} \geq \frac{1}{2}$

Also, $\beta(\theta) = -(\sin^4\theta\cos^4\theta + (1+\cos^2\theta)(1+\sin^2\theta))$

$= -(\sin^4\theta\cos^4\theta + 1 + \cos^2\theta + \sin^2\theta + \sin^2\theta\cos^2\theta)$

$= -(t^2 + t + 2),\ t = \frac{\sin^2 2\theta}{4} \in \left[0, \frac{1}{4}\right]$

$\Rightarrow \beta(\theta) \geq -\frac{37}{16}$

10. Ans. (A, C, D)

Sol. $(\text{adj}M)_{11} = 2 - 3b = -1 \Rightarrow b = 1$

Also, $(\text{adj}M)_{22} = -3a = -6 \Rightarrow a = 2$

Now, $\det M = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2$

$\Rightarrow \det(\text{adj}M^2) = (\det M^2)^2$

$= (\det M)^4 = 16$

Also $M^{-1} = \frac{\text{adj}M}{\det M}$

$\Rightarrow \text{adj}M = -2M^{-1}$

$\Rightarrow (\text{adj}M)^{-1} = \frac{-1}{2}M$

And, $\text{adj}(M^{-1}) = (M^{-1})^{-1}\det(M^{-1})$

$= \frac{1}{\det M}M = \frac{-M}{2}$

Hence, $(\text{adj}M)^{-1} + \text{adj}(M^{-1}) = -M$

Further, MX = b

$\Rightarrow$ $X = M^{-1}b = \frac{-\text{adj}M}{2}b$

$= \frac{-1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \frac{-1}{2}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow (\alpha, \beta, \gamma) = (1, -1, 1)$

11. Ans. (B, C, D)

Sol. Let $Q = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$$X = \sum_{k=1}^{6} \left(P_k Q P_K^T\right)$$

$$X^T = \sum_{k=1}^{6} \left(P_k Q P_K^T\right)^T = X$$

X is symmetric

Let $R = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$XR = \sum_{k=1}^{6} P_K Q P_K^T R \,.\ [\because P_K{}^T R = R]$$

$$= \sum_{K=1}^{6} P_K QR. = \left(\sum_{K=1}^{6} P_K\right) QR$$

$$\sum_{K=1}^{6} P_K = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad QR = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$\Rightarrow XR = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} = 30R$$

$\Rightarrow \alpha = 30.$

$$\text{Trace } X = \text{Trace } \left(\sum_{K=1}^{6} P_K Q P_K^T\right)$$

$$= \sum_{K=1}^{6} \text{Trace}\left(P_K Q P_K^T\right) = 6\left(\text{Trace } Q\right) = 18$$

$$X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 30 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left(X - 30I\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = O \Rightarrow \left|X - 30I\right| = 0$$

$\Rightarrow X - 30I$ is non-invertible

12. Ans. (C, D)

Sol. $\det(R) = \det(PQP^{-1}) = (\det P)(\det Q) \left(\dfrac{1}{\det P}\right)$

$= \det Q$

$= 48 - 4x^2$

Option (A) :

for $x = 1$ $\det(R) = 44 \neq 0$

$\therefore$ for equation $R \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We will have trivial solution

$\alpha = \beta = \gamma = 0$

Option (B) :

$PQ = QP$

$PQP^{-1} = Q$

$R = Q$

No value of x.

Option (C) :

$$\det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$$

$= (40 - 4x^2) + 8 = 48 - 4x^2 = \det R \ \forall\ x \in R$

Option (D):

$$R = \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\left(R - 6I\right) \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = O$$

$$\Rightarrow -4 + a + \frac{2b}{3} = 0$$

$$-2a + \frac{4b}{3} = 0$$

$\Rightarrow a = 2 \qquad b = 3$

$a + b = 5$

13. Ans. (A, D)

Sol. We find D = 0 & since no pair of planes are parallel, so there are infinite number of solutions.

Let $\alpha P_1 + \lambda P_2 = P_3$

$\Rightarrow P_1 + 7P_2 = 13P_3$

$\Rightarrow b_1 + 7b_2 = 13b_3$

(A) $D \neq 0 \Rightarrow$ unique solution for any b_1, b_2, b_3

(B) D = 0 but $P_1 + 7P_2 \neq 13P_3$

(C) As planes are parallel and there exist infinite ordered triplet for which they will be non coincident although satisfying $b_1 + 7b_2 = 13b_3$.

$\therefore$ rejected.

(D) $D \neq 0$

14. Ans. (4)

Sol. $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \underbrace{(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2)}_{x} - \underbrace{(a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)}_{y}$$

Now if $x \leq 3$ and $y \geq -3$

the Δ can be maximum 6

But it is not possible

as x = 3 $\Rightarrow$ each term of x = 1

and y = 3 $\Rightarrow$ each term of y = –1

$\Rightarrow \prod_{i=1}^{3} a_ib_ic_i = 1$ and $\prod_{i=1}^{3} a_ib_ic_i = -1$

which is contradiction

so now next possibility is 4

which is obtained as

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1+1) - 1(-1-1) + 1(1-1) = 4$$

15. Ans. (A, B)

16. Ans. (1)

Sol. $\Delta = 0 \Rightarrow 1(1-\alpha^2) - \alpha(\alpha - \alpha^3) + \alpha^2(\alpha^2 - \alpha^2) = 0$

$(1-\alpha^2) - \alpha^2 + \alpha^4 = 0$

$(\alpha^2 - 1)^2 = 0 \Rightarrow \alpha = \pm 1$

but at $\alpha = 1$ No solution so rejected

at $\alpha = -1$ all three equation become

$x - y + z = 1$ (coincident planes)

$\therefore 1 + \alpha + \alpha^2 = 1$

17. Ans. (A)

Sol. Let $M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\therefore tr(M^TM) = a^2 + b^2 + c^2 + d^2 + c^2 + f^2 + g^2 + h^2 + i^2$

= 5, where entries are {0,1,2}

Only two cases are possible.

(I) five entries 1 and other four zero

$\therefore {}^9C_5 \times 1$

(II) One entry is 2, one entry is 1 and others are 0.

$\therefore {}^9C_2 \times 2!$

Total = 126 + 72 = 198

18. Ans. (B, C)

Sol. PQ = kI

$|P|.|Q| = k^3$

$\Rightarrow |P| = 2k \neq 0 \Rightarrow$ P is an invertible matrix

$\because$ PQ = kI

$\therefore Q = kP^{-1}I$

$\therefore Q = \dfrac{adj.P}{2}$

$\because q_{23} = -\dfrac{k}{8}$

$\therefore \dfrac{-(3\alpha+4)}{2} = -\dfrac{k}{8} \Rightarrow k = 4$

$\therefore |P| = 2k \Rightarrow k = 10 + 6\alpha$ ...(i)

Put value of k in (i).. we get $\alpha = -1$

$\therefore 4\alpha - k + 8 = 0$

& det (P(adj.Q)) = |P| |adj.Q| = $2k.\left(\dfrac{k^2}{2}\right)^2$

$= \dfrac{k^5}{2} = 2^9$

19. Ans. (B)

Sol. $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 16+32 & 8 & 1 \end{bmatrix}$

so, $P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 16+32+48 & 12 & 1 \end{bmatrix}$

(from the symmetry)

$$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ \frac{16.50.51}{2} & 200 & 1 \end{bmatrix}$$

As, $P^{50} - Q = I \Rightarrow q_{31} = \frac{16.50.51}{2}$

$q_{32} = 200$ and $q_{21} = 200$

$\therefore \frac{q_{31} + q_{32}}{q_{21}} = \frac{16.50.51}{2.200} + 1$

$= 102 + 1 = 103$

20. Ans. (C, D)

Sol. $x^T = -x,\ y^T = -y,\ z^T = z$

(A) Let $P = y^3z^4 - z^4y^3$

$P^T = (y^3z^4)^T - (z^4y^3)^T$

$= -z^4y^3 + y^3z^4 = P \Rightarrow$ symmetric

(B) Let $P = x^{44} + y^{44}$

$P^T = (X^{44})^T + (y^{44})^T = P \Rightarrow$ symmetric

(C) Let $P = x^4z^3 - z^3x^4$

$P^T = (z^3)^T(x^4)^T - (x^4)^T(z^3)^T$

$= z^3x^4 - x^4z^3 = -P \Rightarrow$ skew symmetric

(D) Let $P = x^{23} + y^{23}$

$P^T = -x^{23} - y^{23} = -P \Rightarrow$ skew symmetric

21. Ans. (C, D)

Sol. Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(A) Given that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \Rightarrow a = b = c = \alpha$(let)

$\Rightarrow M = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \Rightarrow |M| = 0 \Rightarrow$ Non-invertible

(B) Given that [b c] = [a b] $\Rightarrow a = b = c = \alpha$(let)

again $|M| = 0 \Rightarrow$ Non-invertible

(C) As given $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$

($\because$ a & c are non zero)

$\Rightarrow$ M is invertible

(D) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow |M| = ac - b^2 \neq 0$

$\because$ ac is not equal to square of an integer

$\therefore$ M is invertible

22. Ans. (A, B)

Sol. (A) $(M - N^2)(M + N^2) = \mathbf{O}$ ...(1)

$(\therefore MN^2 = N^2M)$

$\Rightarrow |M - N^2|\,|M + N^2| = 0$

Case I : If $|M + N^2| = 0$

$\therefore |M^2 + MN^2| = 0$

Case II : If $|M + N^2| \neq 0 \Rightarrow M + N^2$ is invertible

from (1)

$(M - N^2)(M + N^2)(M + N^2)^{-1} = \mathbf{O}$

$\Rightarrow M - N^2 = \mathbf{O}$ which is wrong

(B) $(M + N^2)(M - N^2) = \mathbf{O}$

pre-multiply by M

$\Rightarrow (M^2 + MN^2)(M - N^2) = \mathbf{O}$ ...(2)

Let $M - N^2 = U$

$\Rightarrow$ from equation (2) there exist same non zero 'U'

$(M^2 + MN^2)U = \mathbf{O}$