LIMITS

1. Let α be a positive real number. Let $f: \mathbb{R} \to \mathbb{R}$ and $g : (\alpha, \infty) \to \mathbb{R}$ be the functions defined by

$$f(x) = \sin\left(\frac{\pi x}{12}\right) \text{ and } g(x) = \frac{2\log_e\left(\sqrt{x} - \sqrt{\alpha}\right)}{\log_e\left(e^{\sqrt{x}} - e^{\sqrt{\alpha}}\right)}.$$

Then the value of $\lim_{x \to \alpha^+} f(g(x))$ is _______. **[JEE(Advanced) 2022]**

2. If

$$\beta = \lim_{x \to 0} \frac{e^{x^3} - \left(1 - x^3\right)^{\frac{1}{3}} + \left(\left(1 - x^2\right)^{\frac{1}{2}} - 1\right)\sin x}{x \sin^2 x}$$

then the value of 6β is ______. **[JEE(Advanced) 2022]**

3. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x \to 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is ______. **[JEE(Advanced) 2020]**

4. The value of the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(2\sin 2x \sin\frac{3x}{2} + \cos\frac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2}\cos 2x + \cos\frac{3x}{2}\right)}$$

is_______. **[JEE(Advanced) 2020]**

5. Let F : $\mathbb{R} \to \mathbb{R}$ be a function. We say that f has

PROPERTY 1 if $\lim_{h \to 0} \frac{f(h) - f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim_{h \to 0} \frac{f(h) - f(0)}{h^2}$ exists and is finite.

Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $f(x) = x|x|$ has PROPERTY 2
(B) $f(x) = x^{2/3}$ has PROPERTY 1
(C) $f(x) = \sin x$ has PROPERTY 2
(D) $f(x) = |x|$ has PROPERTY 1

6. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1 + (x + j)(x + j - 1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018]**

(A) $\sum_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$

(B) $\sum_{j=1}^{10} \left(1 + f_j'(0)\right)\sec^2\left(f_j(0)\right) = 10$

(C) For any fixed positive integer n, $\lim_{x \to \infty} \tan\left(f_n(x)\right) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x \to \infty} \sec^2\left(f_n(x)\right) = 1$

7. Let f(x) = $\frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x \neq 1$. Then **[JEE(Advanced) 2017]**

(A) $\lim_{x\to 1+} f(x)$ does not exist

(B) $\lim_{x\to 1-} f(x)$ does not exist

(C) $\lim_{x\to 1-} f(x) = 0$

(D) $\lim_{x\to 1+} f(x) = 0$

8. Let $\alpha, \beta \in \mathbb{R}$ be such that $\lim\limits_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x} = 1$. Then $6(\alpha + \beta)$ equals **[JEE(Advanced) 2017]**

9. Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all for $x \in \mathbb{R}$ and and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in \mathbb{R}$. Let $(f\text{o}g)(x)$ denotes $f(g(x))$ and $(g\text{o}f)(x)$ denotes $g(f(x))$. Then which of the following is (are) true?

[JEE(Advanced) 2015]

(A) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(B) Range of $f\text{o}g$ is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(C) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$

(D) There is an $x \in \mathbb{R}$ such that $(g\text{o}f)(x) = 1$

10. Let m and n be two positive integers greater than 1. If

$$\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\frac{e}{2}\right)$$

then the value of $\frac{m}{n}$ is **[JEE(Advanced) 2015]**

11. The largest value of the non-negative integer a for which $\lim\limits_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$ is

[JEE(Advanced) 2014]

SOLUTIONS

1. Ans. (0.50)

Sol. $$\lim_{x\to\alpha^+}\frac{2\ell n\left(\sqrt{x}-\sqrt{\alpha}\right)}{\ell n\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)}\quad\left(\frac{0}{0}\text{form}\right)$$

$\therefore$ Using Lopital rule,

$$=2\lim_{x\to\alpha^+}\frac{\left(\frac{1}{\sqrt{x}-\sqrt{\alpha}}\right).\frac{1}{2\sqrt{x}}}{\left(\frac{1}{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}\right).e^{\sqrt{x}}.\frac{1}{2\sqrt{x}}}$$

$$=\frac{2}{e^{\sqrt{\alpha}}}\lim_{x\to\alpha^+}\frac{\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)}{\left(\sqrt{x}-\sqrt{\alpha}\right)}\quad\left(\frac{0}{0}\right)$$

$$=\frac{2}{e^{\sqrt{\alpha}}}\lim_{x\to\alpha^+}\frac{\left(e^{\sqrt{x}}.\frac{1}{2\sqrt{x}}-0\right)}{\left(\frac{1}{2\sqrt{x}}-0\right)}=2$$

so, $\lim\limits_{x\to\alpha^+}f\left(g(x)\right)=\lim\limits_{x\to\alpha^+}f(2)$

$$=f(2)=\sin\frac{\pi}{6}=\frac{1}{2}=0.50$$

2. Ans. (5)

Sol. $$\beta=\lim_{x\to0}\frac{e^{x^3}-\left(1-x^3\right)^{1/3}}{\frac{x\sin^2x}{x^2}x^2}+\frac{\left(\left(1-x^2\right)^{1/2}-1\right)\sin x}{x\frac{\sin^2x}{x^2}x^2}$$

use expansion

$$\beta=\lim_{x\to0}\frac{\left(1+x^3\right)-\left(1-\frac{x^3}{3}\right)}{x^3}+\lim_{x\to0}\frac{\left(\left(1-\frac{x^2}{2}\right)-1\right)}{x^2}\frac{\sin x}{x}$$

$$\beta=\lim_{x\to0}\frac{4x^3}{3x^3}+\lim_{x\to0}\frac{-x^2}{2x^2}$$

$$\beta=\frac{4}{3}-\frac{1}{2}=\frac{5}{6}$$

$6\beta=5$

3. Ans. (1.00)

Sol. $$\lim_{x-0^+}\frac{e^{\left(\frac{\ell n(1-x)}{x}\right)}-\frac{1}{e}}{x^a}$$

$$=\lim_{x-0^+}\frac{1}{e}\frac{e^{\left(1+\frac{\ell n(1-x)}{x}\right)}-1}{x^a}$$

$$=\frac{1}{e}\lim_{x-0^+}\frac{1+\frac{\ell n(1-x)}{x}}{x^a}$$

$$=\frac{1}{e}\lim_{x-0^+}\frac{\ell n(1-x)+x}{x^{(a+1)}}$$

$$=\frac{1}{e}\lim_{x-0^+}\frac{\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-.......\right)+x}{x^{a+1}}$$

Thus, a = 1

4. Ans. (8)

Sol.

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}\cdot2\sin2x\cos x}{2\sin2x\sin\frac{3x}{2}+\left(\cos\frac{5x}{2}-\cos\frac{3x}{2}\right)-\sqrt{2}(1+\cos2x)}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2x}{2\sin2x\left(\sin\frac{3x}{2}-\sin\frac{x}{2}\right)-2\sqrt{2}\cos^2x}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2x}{4\sin x\cos x\left(2\cos x\cdot\sin\frac{x}{2}\right)-2\sqrt{2}\cos^2x}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x}{8\sin x\cdot\sin\frac{x}{2}-2\sqrt{2}}=8$$

5. Ans. (B, D)

Sol. **P -1 :**

$$\lim_{h\to0}\frac{f(h)-f(0)}{\sqrt{|h|}}=\text{exist and finite}$$

(B) $f(x)=x^{2/3},\lim\limits_{h\to0}\frac{h^{2/3}-0}{\sqrt{|h|}}=\lim\limits_{h\to0}\frac{|h|^{2/3}}{\sqrt{|h|}}=0$

(D) $f(x)=|x|,\lim\limits_{h\to0}\frac{|h|-0}{\sqrt{|h|}}\Rightarrow\lim\limits_{h\to0}\sqrt{|h|}=0$

P-2 :

$$\lim_{h\to0}\frac{f(h)-f(0)}{h^2}=\text{exist and finite}$$

(A) $f(x) = x|x|$,

$$\lim_{h\to0}\frac{h|h|-0}{h^2}=\begin{cases}\text{RHL}=\lim\limits_{h\to0}\frac{h^2}{h^2}=1\\ \text{LHL}=\lim\limits_{h\to0}\frac{-h^2}{h^2}=-1\end{cases}$$

(C) $f(x) = \sin x$ $\lim\limits_{h\to0}\frac{\sin h-0}{h^2}=\text{DNE}$

6. Ans. (D)

Sol. $f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$

$f_n(x) = \sum_{j=1}^{n} [\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$

$f_n(x) = \tan^{-1}(x+n) - \tan^{-1}x$

$f_n'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$

$\therefore$ $\tan(f_n(x)) = \tan[\tan^{-1}(x+n) - \tan^{-1}x]$

$\tan(f_n(x)) = \frac{(x+n)-x}{1+x(x+n)}$

$\tan(f_n(x)) = \frac{n}{1+x^2+nx}$

$\therefore$ $\sec^2(f_n(x)) = 1 + \tan^2(f_n(x))$

$\sec^2(f_n(x)) = 1 + \left(\frac{n}{1+x^2+nx}\right)^2$

$\because$ 0 is not in the domain of f_n

so, no menaning of $f_j'(0)$ and $f_j(0)$

$\therefore$ option (A) and (B) are wrong

(C) For any fixed positive integer n,

$\lim_{x\to\infty} \tan(f_n(x)) = \lim_{x\to\infty} \frac{n}{1+x^2+nx} = 0$

(D) For any fixed positive integer n,

$\lim_{x\to\infty} \sec^2(f_n(x)) = \lim_{x\to\infty}\left(1+\left(\frac{n}{1+x^2+nx}\right)^2\right)$

$= 1$

7. Ans. (A,C)

Sol. $f(x) = \begin{cases} (1-x)\cos\frac{1}{1-x}, & x<1 \\ -(1+x)\cos\frac{1}{1-x}, & x>1 \end{cases}$

$\lim_{x\to 1^+} f(x) = \text{d.n.e}, \ \lim_{x\to 1^-} f(x) = 0$

8. Ans. (7)

Sol. If $\alpha \neq 1$, then $\lim_{x\to 0} \frac{x\sin\beta x}{\frac{\alpha x - \sin x}{x}} = 0$

$\therefore$ $\alpha = 1$ $\Rightarrow \lim_{x\to 0} \frac{\beta x^3 \frac{\sin\beta x}{\beta x}}{x^3\left(\frac{x-\sin x}{x^3}\right)} = \frac{\beta}{1/6}$

$\Rightarrow$ $6\beta = 1$ $\Rightarrow \beta = \frac{1}{6}$

$6(\alpha+\beta) = 7$

9. Ans. (A, B, C)

Sol. (A) $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right), x \in R$

$= \sin\left(\frac{\pi}{6}\sin\theta\right), \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$= \sin\alpha, \ \alpha \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

(B) $f(g(x)) = f(t), t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow f(t) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\lim_{x\to 0} \frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)} \frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}$

$= 1.\frac{\pi}{6}$

(D) $g(f(x)) = 1 \Rightarrow \sin f(x) = \frac{2}{\pi}$

but $f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

$\Rightarrow \sin f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ so no solutions

10. Ans. (2)

Sol. $\lim_{\alpha\to 0} \frac{e\left(e^{\cos\alpha^n - 1} - 1\right)}{\left(\cos\alpha^n - 1\right)} \frac{\left(\cos\alpha^n - 1\right)}{\left(\alpha^n\right)^2} \alpha^{2n-m}$

$= -\frac{e}{2}$ $\quad \therefore 2n = m \Rightarrow \frac{m}{n} = 2$

11. Ans. (0)

Sol. $\lim_{x\to 1}\left\{\frac{\sin(x-1)+a(1-x)}{(x-1)+\sin(x-1)}\right\}^{\frac{(1+\sqrt{x})(1-\sqrt{x})}{1-\sqrt{x}}} = \frac{1}{4}$

$\lim_{x\to 1}\left\{\frac{\frac{\sin(x-1)}{(x-1)} - a}{1+\frac{\sin(x-1)}{(x-1)}}\right\}^{1+\sqrt{x}} = \frac{1}{4} \Rightarrow \left(\frac{1-a}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow (a-1)^2 = 1 \Rightarrow a = 2$ or 0

but for a = 2 base of above limit approaches $-\frac{1}{2}$ and exponent approaches to 2 and since base cannot be negative hence limit does not exist.