

INVERSE TRIGONOMETRIC FUNCTION

1. Let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for $x \in \mathbb{R}$. Then the number of real solutions of the equation

$$\sqrt{1 + \cos(2x)} = \sqrt{2} \tan^{-1}(\tan x) \text{ in the set } \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \text{ is equal to}$$

[JEE(Advanced) 2023]

2. For any $y \in \mathbb{R}$, let $\cot^{-1}(y) \in (0, \pi)$ and $\tan^{-1}(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the sum of all the solutions of the

equation $\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \cot^{-1}\left(\frac{9-y^2}{6y}\right) = \frac{2\pi}{3}$ for $0 < |y| < 3$, is equal to [JEE(Advanced) 2023]

- (A) $2\sqrt{3} - 3$ (B) $3 - 2\sqrt{3}$ (C) $4\sqrt{3} - 6$ (D) $6 - 4\sqrt{3}$

3. Considering only the principal values of the inverse trigonometric functions, the value of

$$\frac{3}{2} \cos^{-1} \sqrt{\frac{2}{2+\pi^2}} + \frac{1}{4} \sin^{-1} \frac{2\sqrt{2}\pi}{2+\pi^2} + \tan^{-1} \frac{\sqrt{2}}{\pi}$$

is _____.

[JEE(Advanced) 2022]

4. For any positive integer n , let $S_n : (0, \infty) \rightarrow \mathbb{R}$ be defined by $S_n(x) = \sum_{k=1}^n \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$, where for any $x \in \mathbb{R}$, $\cot^{-1}x \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are)

TRUE ?

[JEE(Advanced) 2021]

- (A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$ (B) $\lim_{n \rightarrow \infty} \cot(S_n(x)) = x$, for all $x > 0$
 (C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$ (D) $\tan(S_n(x)) \leq \frac{1}{2}$, for all $n \geq 1$ and $x > 0$

5. For non-negative integers n , let

$$f(n) = \frac{\sum_{k=0}^n \sin\left(\frac{k+1}{n+2}\pi\right) \sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^n \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}x$ takes values in $[0, \pi]$, which of the following options is/are correct ?

[JEE(Advanced) 2019]

- (A) $\sin(7 \cos^{-1} f(5)) = 0$ (B) $f(4) = \frac{\sqrt{3}}{2}$
 (C) $\lim_{n \rightarrow \infty} f(n) = \frac{1}{2}$ (D) If $\alpha = \tan(\cos^{-1} f(6))$, then $\alpha^2 + 2\alpha - 1 = 0$

6. The value of $\sec^{-1}\left(\frac{1}{4} \sum_{k=0}^{10} \sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right) \sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals

7. The number of real solutions of the equation

$$\sin^{-1} \left(\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i \right) = \frac{\pi}{2} - \cos^{-1} \left(\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i \right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is _____.

[JEE(Advanced) 2018]

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

8. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$

and $E_2 = \left\{x \in E_1 : \sin^{-1} \left(\log_e \left(\frac{x}{x-1} \right) \right) \text{ is a real number} \right\}$.

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

Let $f : E_1 \rightarrow \mathbb{R}$ be the function defined by $f(x) = \log_e \left(\frac{x}{x-1} \right)$

and $g : E_2 \rightarrow \mathbb{R}$ be the function defined by $g(x) = \sin^{-1} \left(\log_e \left(\frac{x}{x-1} \right) \right)$

[JEE(Advanced) 2018]

LIST-I

- P. The range of f is
- Q. The range of g contains
- R. The domain of f contains
- S. The domain of g is

LIST-II

- 1. $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
- 2. $(0, 1)$
- 3. $\left[-\frac{1}{2}, \frac{1}{2}\right]$
- 4. $(-\infty, 0) \cup (0, \infty)$
- 5. $\left(-\infty, \frac{e}{e-1}\right]$
- 6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is :

- (A) P → 4; Q → 2; R → 1; S → 1
- (C) P → 4; Q → 2; R → 1; S → 6

- (B) P → 3; Q → 3; R → 6; S → 5
- (D) P → 4; Q → 3; R → 6; S → 5

9. If $\alpha = 3 \sin^{-1} \left(\frac{6}{11} \right)$ and $\beta = 3 \cos^{-1} \left(\frac{4}{9} \right)$ where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are)

[JEE(Advanced) 2015]

- (A) $\cos \beta > 0$
- (B) $\sin \beta < 0$
- (C) $\cos(\alpha + \beta) > 0$
- (D) $\cos \alpha < 0$

10. Let $f : [0, 4\pi] \rightarrow [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is

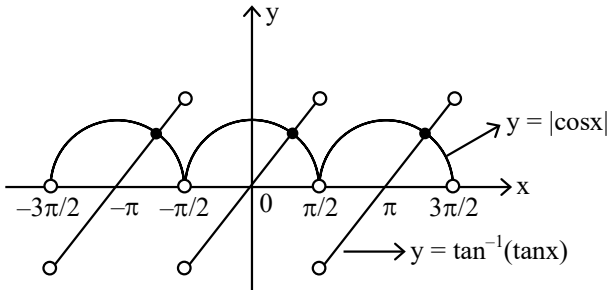
[JEE(Advanced) 2014]

SOLUTIONS

1. Ans. (3)

Sol. $\sqrt{2}|\cos x| = \sqrt{2} \cdot \tan^{-1}(\tan x)$

$|\cos x| = \tan^{-1} \tan x$



No. of solutions = 3

2. Ans. (C)

Sol. Case-I : $y \in (-3, 0)$

$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \pi + \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$

$2 \tan^{-1}\left(\frac{6y}{9-y^2}\right) = -\frac{\pi}{3}$

$y^2 - 6\sqrt{3}y - 9 = 0 \Rightarrow y = 3\sqrt{3} - 6 (\because y \in (-3, 0))$

Case-II : $y \in (0, 3)$

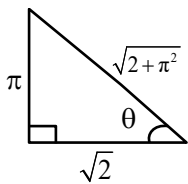
$2 \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3} \Rightarrow \sqrt{3}y^2 + 6y - 9\sqrt{3} = 0$

$y = \sqrt{3}$ or $y = -3\sqrt{3}$ (rejected)

sum = $\sqrt{3} + 3\sqrt{3} - 6 = 4\sqrt{3} - 6$

3. Ans. (2.35 or 2.36)

Sol. $\cos^{-1} \frac{2}{\sqrt{2+\pi^2}} = \tan^{-1} \frac{\pi}{\sqrt{2}}$



$\sin^{-1}\left(\frac{2\sqrt{2}\pi}{2+\pi^2}\right) = \sin^{-1}\left(\frac{2 \times \frac{\pi}{\sqrt{2}}}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2}\right)$

$= \pi - 2 \tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$

(As, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2 \tan^{-1} x, x \geq 1$)

and $\tan^{-1} \frac{\sqrt{2}}{\pi} = \cot^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$

∴ Expression

$= \frac{3}{2} \left(\tan^{-1} \frac{\pi}{\sqrt{2}}\right) + \frac{1}{4} \left(\pi - 2 \tan^{-1} \frac{\pi}{\sqrt{2}}\right) + \cot^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$

$= \left(\frac{3}{2} - \frac{2}{4}\right) \tan^{-1} \frac{\pi}{\sqrt{2}} + \frac{\pi}{4} + \cot^{-1} \frac{\pi}{\sqrt{2}}$

$= \left(\tan^{-1} \frac{\pi}{\sqrt{2}} + \cot^{-1} \frac{\pi}{\sqrt{2}}\right) + \frac{\pi}{4}$

$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} = 2.35 \text{ or } 2.36$

4. Ans. (A, B)

Sol. $S_n(x) = \sum_{k=1}^n \tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$

$= \sum_{k=1}^n \tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$

$S_n(x) = \tan^{-1}(nx+x) - \tan^{-1}x$

$= \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$

(A) $S_{10}(x) = \tan^{-1} \frac{10x}{1+11x^2}$
 $= \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right) \quad (x > 0)$

(B) $\lim_{n \rightarrow \infty} \cot(S_n(x)) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \left(1 + \frac{1}{n}\right)x^2}{x}$
 $= x \quad (x > 0)$

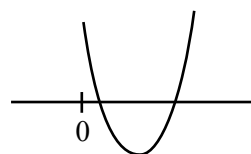
(C) $S_3(x) = \tan^{-1} \frac{3x}{1+4x^2} = \frac{\pi}{4}$
 $\Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin \mathbb{R}$

(D) $\tan(S_n(x)) = \frac{nx}{1+(n+1)x^2}; \forall n \geq 1; x > 0$

We need to check the validity of

$\frac{nx}{1+(n+1)x^2} \leq \frac{1}{2} \quad \forall n \geq 1; x > 0; n \in \mathbb{N}$

$\Rightarrow 2nx \leq (n+1)x^2 + 1$



$\Rightarrow (n+1)x^2 - 2nx + 1 \geq 0 \quad \forall n \geq 1; x > 0; n \in \mathbb{N}$

Discriminant of $y = (n + 1)x^2 - 2nx + 1$ is

$$D = 4n^2 - 4(n + 1) \text{ and } n \in \mathbb{N}$$

$D < 0$ for $n = 1$; true for $x > 0$

$D > 0$ for $n \geq 2 \Rightarrow \exists$ some $x > 0$

for which $y < 0$ as both roots of $y = 0$ will be positive.

$$y = (n + 1)x^2 - 2nx + 1, n \geq 2$$

So, $y \geq 0 \forall n \geq 1 ; \forall x > 0 ; n \in \mathbb{N}$ is false.

5. **Ans. (A, B, D)**

$$\text{Sol. } f(n) = \frac{\sum_{k=0}^n \left(\cos\left(\frac{\pi}{n+2}\right) - \cos\left(\frac{2k+3}{n+2}\right)\pi \right)}{\sum_{k=0}^n \left(1 - \cos\left(\frac{2k+2}{n+2}\right)\pi \right)}$$

$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) - \left(\sum_{k=0}^n \cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{(n+1) - \left(\sum_{k=0}^n \cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n) = \frac{(n+1)\cos\frac{\pi}{n+2} - \left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)} \cdot \cos\left(\frac{n+3}{n+2}\right)\pi\right)}{(n+1) - \left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)} \cdot \cos\left(\frac{2(n+2)\pi}{2(n+2)}\right)\right)}$$

$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{(n+1) + 1}$$

$$\Rightarrow g(x) = \cos\left(\frac{\pi}{n+2}\right)$$

$$\Rightarrow g(x) = \cos\left(\frac{\pi}{n+2}\right)$$

$$(A) \sin\left(7\cos^{-1}\cos\frac{\pi}{7}\right) = \sin\pi = 0$$

$$(B) f(4) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$(C) \lim_{n \rightarrow \infty} \cos\left(\frac{\pi}{n+2}\right) = 1$$

$$(D) \alpha = \tan\left(\cos^{-1}\cos\frac{\pi}{8}\right) = \sqrt{2} - 1 \Rightarrow \alpha + 1 = \sqrt{2}$$

$$\alpha^2 + 2\alpha - 1 = 0$$

6. **Ans. (0.00)**

$$\text{Sol. } \sec^{-1}\left(\frac{1}{4} \sum_{k=0}^{10} \frac{1}{\cos\left(\frac{7\pi}{12} + \frac{k\pi}{12}\right) \cos\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4} \sum_{k=0}^{10} \frac{\sin\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2} - \left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\right)}{\cos\left(\frac{7\pi}{12} + \frac{k\pi}{12}\right) \cdot \cos\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4} \left(\sum_{k=0}^{10} \tan\left(\frac{7\pi}{12} + (k+1)\frac{\pi}{2}\right) - \tan\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4} \left(\tan\left(\frac{11\pi}{2} + \frac{7\pi}{12}\right) - \tan\left(\frac{7\pi}{12}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4} \left(-\cot\frac{7\pi}{12} - \tan\frac{7\pi}{12}\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4} \left(-\frac{1}{\sin\frac{7\pi}{12} \cos\frac{7\pi}{12}}\right)\right)$$

$$= \sec^{-1}\left(-\frac{1}{2} \times \frac{1}{\sin\frac{7\pi}{6}}\right) = \sec^{-1}(1) = 0.00$$

7. **Ans. (2)**

$$\text{Sol. } \sum_{i=1}^{\infty} x^{i+1} = \frac{x^2}{1-x}$$

$$\sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \frac{x}{2-x}$$

$$\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i = \frac{-x}{2+x}$$

$$\sum_{i=1}^{\infty} (-x)^i = \frac{-x}{1+x}$$

To have real solutions

$$\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty} \left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i$$

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{-x}{2+x} + \frac{x}{1+x}$$

$$x(x^3 + 2x^2 + 5x - 2) = 0$$

$$\therefore x = 0 \text{ and let } f(x) = x^3 + 2x^2 + 5x - 2$$

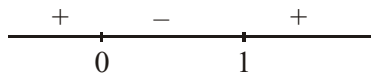
$$f(x) > 0 \Rightarrow f \text{ is } \uparrow$$

$$f\left(\frac{1}{2}\right) \cdot f\left(-\frac{1}{2}\right) < 0$$

Hence two solutions exist

8. Ans. (A)

Sol. $E_1: \frac{x}{x-1} > 0$



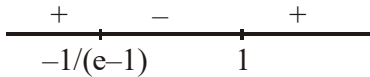
$\Rightarrow E_1: x \in (-\infty, 0) \cup (1, \infty)$

$E_2: -1 \leq \ln\left(\frac{x}{x+1}\right) \leq 1$

$\frac{1}{e} \leq \frac{x}{x+1} \leq e$

Now $\frac{x}{x+1} - \frac{1}{e} \geq 0$

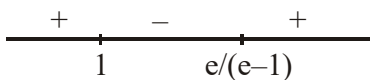
$\Rightarrow \frac{(e-1)x+1}{e(x-1)} \geq 0$



$\Rightarrow x \in \left(-\infty, \frac{1}{1-e}\right] \cup (1, \infty)$

also $\frac{x}{x-1} - e \leq 0$

$\frac{(e-1)x-e}{x-1} \geq 0$



$\Rightarrow x \in (-\infty, 1) \cup \left[\frac{e}{e-1}, \infty\right)$

So $E_2: \left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$

as Range $\frac{x}{x-1}$ of is $\mathbb{R}^+ - \{1\}$

\Rightarrow Range of f is $\mathbb{R} - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Range of g is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ or

$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Now $P \rightarrow 4, Q \rightarrow 2, R \rightarrow 1, S \rightarrow 1$

9. Ans. (B, C, D)

Sol. $\alpha = 3 \sin^{-1} \frac{6}{11}$ & $\beta = 3 \cos^{-1} \frac{4}{9}$

$\therefore \frac{6}{11} > \frac{1}{2} \Rightarrow \sin^{-1} \frac{6}{11} > \sin^{-1} \frac{1}{2}$

$\Rightarrow 3 \sin^{-1} \frac{6}{11} > 3 \sin^{-1} \frac{1}{2} = \frac{\pi}{2}$

$\therefore \alpha > \frac{\pi}{2}$

$\therefore \cos \alpha < 0$

Now, $\beta = 3 \cos^{-1} \frac{4}{9}$

$\therefore \frac{4}{9} < \frac{1}{2} \Rightarrow 3 \cos^{-1} \frac{4}{9} > 3 \cos^{-1} \frac{1}{2}$

$\therefore \beta > \pi$

$\therefore \cos \beta < 0$ & $\sin \beta < 0$

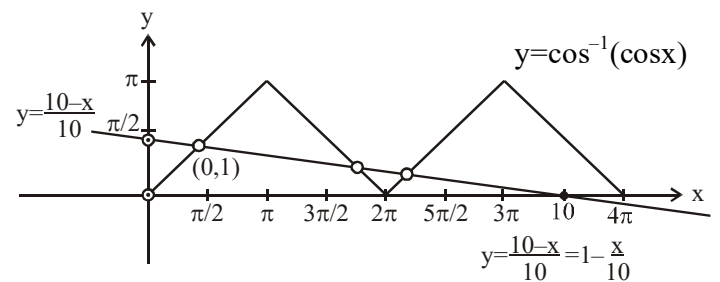
Now, α is slightly greater than $\frac{\pi}{2}$ & β is

slightly greater than π

$\therefore \cos(\alpha + \beta) > 0$

10. Ans. (3)

Sol.



from above figure it is clear that $y = \frac{10-x}{10}$ and

$y = \cos^{-1}(\cos x)$ intersect at 3 distinct points, so

number of solutions = 3