INDEFINITE INTEGRATION

1. Let b be a nonzero real number. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that $f(0) = 1$. If the derivative f' of f satisfies the equation $f'(x) = \frac{f(x)}{b^2 + x^2}$ for all $x \in \mathbb{R}$, then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) If $b > 0$, then f is an increasing function

(B) If $b < 0$, then f is a decreasing function

(C) $f(x)\, f(-x) = 1$ for all $x \in \mathbb{R}$

(D) $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$

SOLUTIONS

1. Ans. (A, C)

Sol. $f'(x)=\dfrac{f(x)}{b^2+x^2}$

$\int \dfrac{f'(x)}{f(x)}dx=\int \dfrac{dx}{x^2+b^2}$

$\Rightarrow \ell n|f(x)|=\dfrac{1}{b}\tan^{-1}\left(\dfrac{x}{b}\right)+c$

Now $f(0) = 1$

$\therefore c = 0$

$\therefore |f(x)|=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\Rightarrow f(x)=\pm e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

since $f(0) = 1$

$\therefore f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$x \to -x$

$f(-x)=e^{-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\therefore f(x).f(-x) = e^0 = 1$ (option C)

and for $b > 0$

$f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\Rightarrow f(x)$ is increasing for all $x \in R$ (option A)