HYPERBOLA

1. Consider the hyperbola

$$\frac{x^2}{100}-\frac{y^2}{64}=1$$

with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1 = \alpha$, with $\alpha < \frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola, intersects the straight line S_1P at P_1. Let δ be the distance of P from the straight line SP_1, and $\beta = S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9}\sin\frac{\alpha}{2}$ is______. **[JEE(Advanced) 2022]**

2. Let a and b be positive real numbers such that a > 1 and b < a. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Suppose the tangent to the hyperbola at P passes through the point (1,0), and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE ?

[JEE(Advanced) 2020]

(A) $1 < e < \sqrt{2}$ (B) $\sqrt{2} < e < 2$ (C) $\Delta = a^4$ (D) $\Delta = b^4$

3. Let H : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

[JEE(Advanced) 2018]

LIST-I	**LIST-II**
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is :

(A) $P \to 4$; $Q \to 2$, $R \to 1$; $S \to 3$

(B) $P \to 4$; $Q \to 3$; $R \to 1$; $S \to 2$

(C) $P \to 4$; $Q \to 1$, $R \to 3$; $S \to 2$

(D) $P \to 3$; $Q \to 4$; $R \to 2$; $S \to 1$

4. If $2x - y + 1 = 0$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE(Advanced) 2017]**

(A) 2a, 4, 1 (B) 2a, 8, 1 (C) a, 4, 1 (D) a, 4, 2

MATCHING TYPE : (Q.5 to Q.7)

Column 1, 2 and 3 contain conics, equation of tangents to the conics and points of contact, respectively.

Column-1	Column-2	Column-3
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

5. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**

(A) (II) (iii) (R) (B) (IV) (iv) (S) (C) (IV) (iii) (S) (D) (II) (iv) (R)

6. If a tangent to a suitable conic (Column 1) is found to be y = x + 8 and its point of contact is (8,16), then which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**

(A) (III) (i) (P) (B) (III) (ii) (Q)

(C) (II) (iv) (R) (D) (I) (ii) (Q)

7. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1,1), then which of the following options is the only **CORRECT** combination for obtaining its equation ?

[JEE(Advanced) 2017]

(A) (II) (ii) (Q) (B) (III) (i) (P)

(C) (I) (i) (P) (D) (I) (ii) (Q)

8. Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

[JEE(Advanced) 2015]

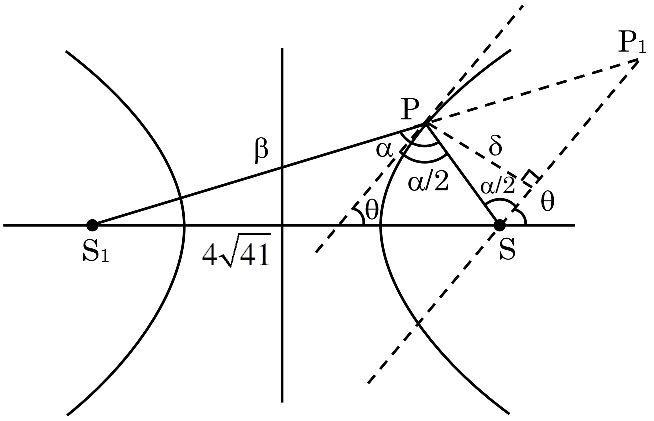
(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$ (B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$ (D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

SOLUTIONS

1. Ans. (7)

Sol.



$S_1P - SP = 20$

$$\beta - \frac{\delta}{\sin\frac{\alpha}{2}} = 20$$

$$\beta^2 + \frac{\delta^2}{\sin^2\frac{\alpha}{2}} - 400 = \frac{2\beta\delta}{\sin\frac{\alpha}{2}}$$

$$\frac{1}{SP} = \frac{\sin\frac{\alpha}{2}}{\delta}$$

$$\cos\alpha = \frac{SP^2 + \beta^2 - 656}{2\beta\frac{\delta}{\sin\frac{\alpha}{2}}}$$

$$= \frac{\frac{2\beta\delta}{\sin\frac{\alpha}{2}} - 256}{\frac{2\beta S}{\sin\frac{\alpha}{2}}} = \cos\alpha$$

$$\frac{\lambda - 128}{\lambda} = \cos\alpha$$

$\lambda(1 - \cos\alpha) = 128$

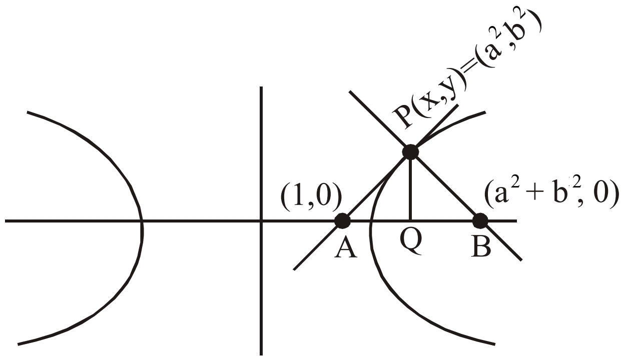
$$\frac{\beta\delta}{\sin\frac{\alpha}{2}}.2\sin^2\frac{\alpha}{2} = 128$$

$$\frac{\beta\delta}{9}\sin\frac{\alpha}{2} = \frac{64}{9} \Rightarrow \left[\frac{\beta\delta}{9}\sin\frac{\alpha}{2}\right] = 7$$

where [.] denotes greatest integer function

2. Ans. (A, D)

Sol.



Since Normal at point P makes equal intercept on co-ordinate axes, therefore slope of Normal = –1

Hence slope of tangent = 1

Equation of tangent

$y - 0 = 1(x - 1)$

$y = x - 1$

Equation of tangent at (x_1y_1)

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$x - y = 1$ (equation of Tangent)

on comparing $x_1 = a^2$, $y_1 = b^2$

Also $a^2 - b^2 = 1$ ...(1)

Now equation of normal at $(x_1,y_1) = (a^2, b^2)$

$y - b^2 = -1(x - a^2)$

$x + y = a^2 + b^2$ ...(Normal)

point of intersection with x-axis is $(a^2 + b^2)$

Now $e = \sqrt{1 + \frac{b^2}{a^2}}$

$$e = \sqrt{1 + \frac{b^2}{b^2+1}} \quad \left[\text{from}\,(1)\ \frac{b^2}{b^2+1} < 1\right]$$

$1 < e < \sqrt{2}$ option (A)

$$\Delta = \frac{1}{2}.AB.PQ$$

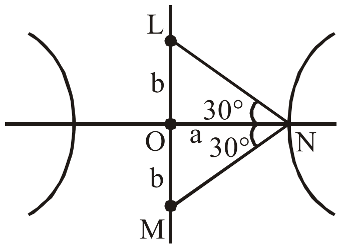
and $\Delta = \frac{1}{2}\left(a^2 + b^2 - 1\right).b^2$

$\Delta = \frac{1}{2}\left(2b^2\right)b^2$ (from (1) $a^2 - 1 = b^2$)

$\Delta = b^4$ so option (D)

3. **Ans. (B)**

Sol.



$\tan 30° = \frac{b}{a}$

$\Rightarrow \quad a = b\sqrt{3}$

Now area of $\Delta LMN = \frac{1}{2}.2b.b\sqrt{3}$

$4\sqrt{3} = \sqrt{3}b^2$

$\Rightarrow \quad b = 2 \ \& \ a = 2\sqrt{3}$

$\Rightarrow \quad e = \sqrt{1+\frac{b^2}{a^2}} = \frac{2}{\sqrt{3}}$

P. Length of conjugate axis = 2b = 4

So $P \to 4$

Q. Eccentricity $e = \frac{2}{\sqrt{3}}$

So $Q \to 3$

R. Distance between foci = 2ae

$= 2(2\sqrt{3})\left(\frac{2}{\sqrt{3}}\right) = 8$

So $R \to 1$

S. Length of latus rectum =

$\frac{2b^2}{a} = \frac{2(2)^2}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

So $S \to 2$

4. **Ans. (B, C, D)**

Sol. The line y = mx + c is tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = a^2m^2 - b^2$

$\therefore (1)^2 = 4a^2 - 16 \Rightarrow a^2 = \frac{17}{4}$

$\Rightarrow a = \frac{\sqrt{17}}{2}$

For option (A), sides are $\sqrt{17}, 4, 1$

($\Rightarrow$ Right angled triangle)

For option (B), sides are $\sqrt{17}, 8, 1$

($\Rightarrow$ Triangle is not possible)

For option (C), sides are $\frac{\sqrt{17}}{2}, 4, 1$

($\Rightarrow$ Triangle is not possible)

For option (D), sides are $\frac{\sqrt{17}}{2}, 4, 2$

($\Rightarrow$ Triangle exist but not right angled)

5. **Ans. (D)**

Sol. $P\left(\sqrt{3}, \frac{1}{2}\right)$; tangent $\sqrt{3}x + 2y = 4$

$\Rightarrow \left(\sqrt{3}\right)x + 4\left(\frac{1}{2}\right)y = 4$ comparing with (II)

$\Rightarrow a = 2 \quad \therefore y = mx + \sqrt{a^2m^2+1}$ is tangent for $m = -\frac{\sqrt{3}}{2}$ i.e (ii)

$\therefore$ point of contact for a = 2, $m = -\frac{\sqrt{3}}{2}$ is R

6. **Ans. (A)**

Sol. y = x + 8 is tangent $\Rightarrow$ m = 1; P(8, 16)

Comparing tangent with (i) of column 2, m = 1 satisfied and a = 8 obtained which matches for point of contact (P) of column 3 and (III) of column 1.

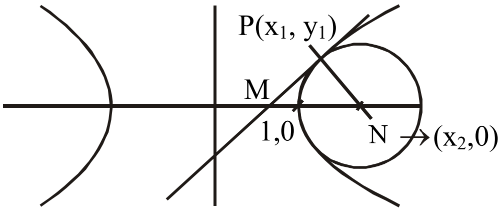
7. **Ans. (D)**

Sol. For $a = \sqrt{2}$ and point (–1,1) only I of column-1 satisfies. Hence equaiton of tangent is –x + y = 2 or y = x + 2 $\Rightarrow$ m = 1 which matches with (ii) of column-2 and also with Q of column-3

Let $f(x) = x + \log_e x - x \log_e x$, $x \in (0,\infty)$.

8. Ans. (A, B, D)

Sol. Given H : $x^2 - y^2 = 1$



Now, equation of family of circle touching hyperbola at (x_1, y_1) is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(xx_1 - yy_1 - 1) = 0$

Now, its centre is $(x_2, 0)$

$\therefore \quad 2y_1 + \lambda y_1 = 0 \quad \Rightarrow \quad \lambda = -2$

$$\therefore \quad x_2 = \frac{2x_1 - \lambda x_1}{2} = \frac{4x_1}{2} = 2x_1$$

$$\therefore \quad P \equiv (x_1, \sqrt{x_1^2 - 1})$$

$$N \equiv (2x_1, 0)$$

$$\& \quad M \equiv \left(\frac{1}{x_1}, 0\right)$$

$$\therefore \quad \ell = \frac{3x_1 + \frac{1}{x_1}}{3} = x_1 + \frac{1}{3x_1}$$

$$\Rightarrow \quad \frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2} \qquad x_1 > 1$$

$$m = \frac{\sqrt{x_1^2 - 1}}{3} \quad \Rightarrow \quad \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$$

$$\text{Also } m = \frac{y_1}{3} \quad \Rightarrow \quad \frac{dm}{dy_1} = \frac{1}{3} \qquad y_1 > 0$$