FUNCTION

1. Let $|M|$ denote the determinant of a square matrix M. Let $g : \left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$ be the function defined by

$$g(\theta) = \sqrt{f(\theta) - 1} + \sqrt{f\left(\dfrac{\pi}{2} - \theta\right) - 1}$$

where

$$f(\theta) = \dfrac{1}{2} \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} + \begin{vmatrix} \sin\pi & \cos\left(\theta + \dfrac{\pi}{4}\right) & \tan\left(\theta - \dfrac{\pi}{4}\right) \\ \sin\left(\theta - \dfrac{\pi}{4}\right) & -\cos\dfrac{\pi}{2} & \log_e\left(\dfrac{4}{\pi}\right) \\ \cot\left(\theta + \dfrac{\pi}{4}\right) & \log_e\left(\dfrac{\pi}{4}\right) & \tan\pi \end{vmatrix}.$$

Let $p(x)$ be a quadratic polynomial whose roots are the maximum and minimum values of the function $g(\theta)$, and $p(2) = 2 - \sqrt{2}$. Then, which of the following is/are TRUE ?   **[JEE(Advanced) 2022]**

(A) $p\left(\dfrac{3 + \sqrt{2}}{4}\right) < 0$ (B) $p\left(\dfrac{1 + 3\sqrt{2}}{4}\right) > 0$

(C) $p\left(\dfrac{5\sqrt{2} - 1}{4}\right) > 0$ (D) $p\left(\dfrac{5 - \sqrt{2}}{4}\right) < 0$

2. If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE** ?

                                                             [JEE(Advanced) 2020]

(A) f is one-one, but **NOT** onto   (B) f is onto, but **NOT** one-one

(C) f is **BOTH** one-one and onto   (D) f is **NEITHER** one-one **NOR** onto

3. Let the function $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \dfrac{4^x}{4^x + 2}$

Then the value of $f\left(\dfrac{1}{40}\right) + f\left(\dfrac{2}{40}\right) + f\left(\dfrac{3}{40}\right) + \ldots + f\left(\dfrac{39}{40}\right) - f\left(\dfrac{1}{2}\right)$ is____.   **[JEE(Advanced) 2020]**

4. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value

of $\dfrac{1}{5!}(\beta - \alpha)$ is _____ .   **[JEE(Advanced) 2018]**

5. Let $f : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ be given by $f(x) = (\log (\sec x + \tan x))^3$ Then :-   **[JEE(Advanced)-2014]**

(A) $f(x)$ is an odd function   (B) $f(x)$ is a one-one function

(C) $f(x)$ is an onto function   (D) $f(x)$ is an even function

<div style="text-align:center">**SOLUTIONS**</div>

1. **Ans. (A, C)**

Sol. $f(\theta) = \dfrac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} +$

$\begin{vmatrix} \sin\pi & \cos\left(\theta+\dfrac{\pi}{4}\right) & \tan\left(\theta-\dfrac{\pi}{4}\right) \\ \sin\left(\theta-\dfrac{\pi}{4}\right) & -\cos\dfrac{\pi}{2} & \log_e\left(\dfrac{4}{\pi}\right) \\ \cot\left(\theta+\dfrac{\pi}{4}\right) & \log_e\dfrac{\pi}{4} & \tan\pi \end{vmatrix}$

$f(\theta) = \dfrac{1}{2}\begin{vmatrix} 2 & \sin\theta & 1 \\ 0 & 1 & \sin\theta \\ 0 & -\sin\theta & 1 \end{vmatrix} +$

$\begin{vmatrix} 0 & -\sin\left(\theta-\dfrac{\pi}{4}\right) & \tan\left(\theta-\dfrac{\pi}{4}\right) \\ \sin\left(\theta-\dfrac{\pi}{4}\right) & 0 & \log_e\left(\dfrac{4}{\pi}\right) \\ -\tan\left(\theta-\dfrac{\pi}{4}\right) & -\log_e\left(\dfrac{4}{\pi}\right) & 0 \end{vmatrix}$

$f(\theta) = (1 + \sin^2\theta) + 0$ (skew symmetric)

$g(\theta) = \sqrt{f(\theta)-1} + \sqrt{f\left(\dfrac{\pi}{2}-\theta\right)-1}$

$= |\sin\theta| + |\cos\theta| \quad$ for $\theta \in \left[0, \dfrac{\pi}{2}\right]$

$g(\theta) \in \left[1, \sqrt{2}\right]$

Again let $P(x) = k\left(x-\sqrt{2}\right)(x-1)$

$2 - \sqrt{2} = k\left(2-\sqrt{2}\right)(2-1)$

$\Rightarrow k = 1 \quad$ ($P(2) = 2-\sqrt{2}$ given)

$\therefore P(x) = \left(x-\sqrt{2}\right)(x-1)$

for option (A) $P\left(\dfrac{3+\sqrt{2}}{4}\right) < 0$ correct

option (B) $P\left(\dfrac{1+3\sqrt{2}}{4}\right) < 0$ incorrect

option (C) $P\left(\dfrac{5\sqrt{2}-1}{4}\right) > 0$ correct

option (D) $P\left(\dfrac{5-\sqrt{2}}{4}\right) > 0$ incorrect

2. **Ans. (C)**

Sol. $f(x)$ is a non-periodic, continuous and odd function

$f(x) = \begin{cases} -x^2 + x\sin x, & x < 0 \\ x^2 - x\sin x, & x \geq 0 \end{cases}$

$f(-\infty) = \underset{x\to-\infty}{\text{Lt}} (-x^2)\left(1 - \dfrac{\sin x}{x}\right) = -\infty$

$f(\infty) = \underset{x\to\infty}{\text{Lt}} x^2\left(1 - \dfrac{\sin x}{x}\right) = \infty$

$\Rightarrow$ Range of $f(x) = R$

$\Rightarrow$ $f(x)$ is an onto function ...(1)

$f'(x) = \begin{cases} -2x + \sin x + x\cos x, & x < 0 \\ 2x - \sin x - x\cos x, & x \geq 0 \end{cases}$

For $(0, \infty)$

$f'(x) = (x - \sin x) + x(1 - \cos x)$

$\qquad$ always +ve always +ve
$\qquad$ or 0 $\qquad$ or 0

$\Rightarrow$ $f'(x) > 0$

$\Rightarrow$ $f'(x) \geq 0, \ \forall \, x \in (-\infty, \infty)$

$\qquad$ equality at x = 0

$\Rightarrow$ $f(x)$ is one-one function ... (2)

From (1) & (2), $f(x)$ is both one-one & onto.

3. **Ans. (19.00)**

Sol. $f(x) + f(1-x) = \dfrac{4^x}{4^x+2} + \dfrac{4^{1-x}}{4^{1-x}+2}$

$= \dfrac{4^x}{4^x+2} + \dfrac{4/4^x}{\dfrac{4}{4^x}+2}$

$= \dfrac{4^x}{4^x+2} + \dfrac{4}{4+2\cdot 4^x}$

$= \dfrac{4^x}{4^x+2} + \dfrac{2}{2+4^x} = 1$

so, $f\left(\dfrac{1}{40}\right) + f\left(\dfrac{2}{40}\right) + ... + f\left(\dfrac{39}{40}\right) - f\left(\dfrac{1}{2}\right)$

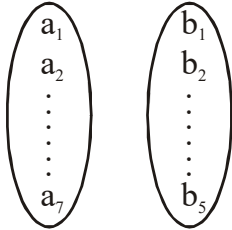$= 19 + f\left(\dfrac{1}{2}\right) - f\left(\dfrac{1}{2}\right) = 19$

4. **Ans. (119)**

Sol. $n(X) = 5$

$n(Y) = 7$

$\alpha \rightarrow$ Number of one-one function $= {}^{7}C_5 \times 5!$

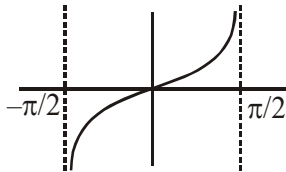$\beta \rightarrow$ Number of onto function Y to X

$$1, 1, 1, 1, 3 \qquad 1, 1, 1, 2, 2$$

$$\frac{7!}{3!4!} \times 5! + \frac{7!}{(2!)^3 3!} \times 5!$$

$$= \left({}^{7}C_3 + 3 \cdot {}^{7}C_3 \right) 5! = 4 \times {}^{7}C_3 \times 5!$$

$$\frac{\beta - \alpha}{5!} = 4 \times {}^{7}C_3 - {}^{7}C_5 = 4 \times 35 - 21 = 119$$

5. **Ans. (A, B, C)**

Sol. $f(x) = \left(\ell n (\sec x + \tan x) \right)^3$

$$f'(x) = \frac{3 \left(\ell n (\sec x + \tan x) \right)^2 \left(\sec x \tan x + \sec^2 x \right)}{(\sec x + \tan x)} > 0$$

$f(x)$ is an increasing function

$$\lim_{x \to -\frac{\pi}{2}} f(x) \to -\infty \quad \& \quad \lim_{x \to \frac{\pi}{2}} f(x) \to \infty$$

Range of $f(x)$ is R and onto function

$$f(-x) = \left(\ell n (\sec x - \tan x) \right)^3 = \left(\ell n \left(\frac{1}{\sec x + \tan x} \right) \right)^3$$

$f(-x) = -(\ell n(\sec x + \tan x))^3$

$f(x) + f(-x) = 0 \Rightarrow f(x)$ is an odd function.