ELLIPSE

1. Let T_1 and T_2 be two distinct common tangents to the ellipse $E : \frac{x^2}{6}+\frac{y^2}{3}=1$ and the parabola $P : y^2 = 12x$. Suppose that the tangent T_1 touches P and E at the point A_1 and A_2, respectively and the tangent T_2 touches P and E at the points A_4 and A_3, respectively. Then which of the following statements is(are) true? **[JEE(Advanced) 2023]**

(A) The area of the quadrilateral $A_1A_2A_3A_4$ is 35 square units

(B) The area of the quadrilateral $A_1A_2A_3A_4$ is 36 square units

(C) The tangents T_1 and T_2 meet the x-axis at the point (–3, 0)

(D) The tangents T_1 and T_2 meet the x-axis at the point (–6, 0)

2. Consider the ellipse

$$\frac{x^2}{4}+\frac{y^2}{3}=1.$$

Let $H(\alpha, 0)$, $0 < \alpha < 2$, be a point. A straight line drawn through H parallel to the y-axis crosses the ellipse and its auxiliary circle at points E and F respectively, in the first quadrant. The tangent to the ellipse at the point E intersects the positive x-axis at a point G. Suppose the straight line joining F and the origin makes an angle ϕ with the positive x-axis.

List-I		List-II	
(I)	If $\phi=\frac{\pi}{4}$, then the area of the triangle FGH is	(P)	$\frac{\left(\sqrt{3}-1\right)^4}{8}$
(II)	If $\phi=\frac{\pi}{3}$, then the area of the triangle FGH is	(Q)	1
(III)	If $\phi=\frac{\pi}{6}$, then the area of the triangle FGH is	(R)	$\frac{3}{4}$
(IV)	If $\phi=\frac{\pi}{12}$, then the area of the triangle FGH is	(S)	$\frac{1}{2\sqrt{3}}$
		(T)	$\frac{3\sqrt{3}}{2}$

The correct option is: **[JEE(Advanced) 2022]**

(A) (I) → (R); (II) → (S); (III) → (Q); (IV) → (P)

(B) (I) → (R); (II) → (T); (III) → (S); (IV) → (P)

(C) (I) → (Q); (II) → (T); (III) → (S); (IV) → (P)

(D) (I) → (Q); (II) → (S); (III) → (Q); (IV) → (P)

3. Let E be the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. For any three distinct points P, Q and Q' on E, let M (P, Q) be the mid-point of the line segment joining P and Q, and M (P, Q') be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between M(P, Q) and M(P, Q'), as P, Q and Q' vary on E, is ____. **[JEE(Advanced) 2021]**

4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

[JEE(Advanced) 2020]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

5. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) The eccentricities of E_{18} and E_{19} are NOT equal

(B) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(C) The length of latus rectum of E_9 is $\frac{1}{6}$

(D) $\sum_{n=1}^{N}(\text{area of } R_n) < 24$, for each positive integer N

6. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{4\sqrt{2}}(\pi-2)$

(D) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi-2)$

Paragraph for Question No. 7 and 8

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

7. The orthocentre of the triangle F_1MN is- **[JEE(Advanced) 2016]**

(A) $\left(-\frac{9}{10},0\right)$ (B) $\left(\frac{2}{3},0\right)$ (C) $\left(\frac{9}{10},0\right)$ (D) $\left(\frac{2}{3},\sqrt{6}\right)$

8. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is-

[JEE(Advanced) 2016]

(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3

9. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2,0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0,0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is **[JEE(Advanced) 2015]**

10. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are)

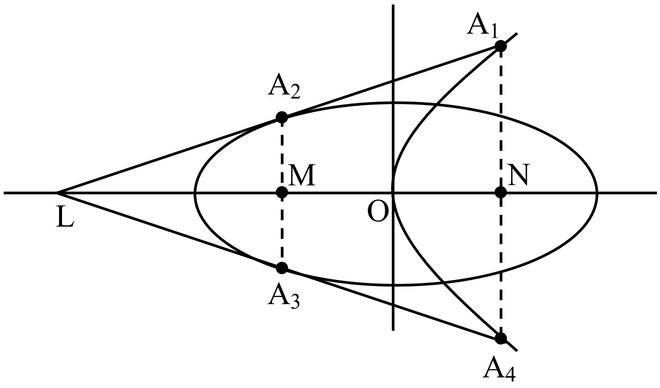
[JEE(Advanced) 2015]

(A) $e_1^2+e_2^2=\frac{43}{40}$ (B) $e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}}$ (C) $\left|e_1^2-e_2^2\right|=\frac{5}{8}$ (D) $e_1e_2=\frac{\sqrt{3}}{4}$

SOLUTIONS

1. Ans. (A, C)

Sol



$y = mx + \frac{3}{m}$

$C^2 = a^2m^2 + b^2$

$\frac{9}{m^2} = 6m^2 + 3 \quad \Rightarrow m^2 = 1$

T_1 & T_2

$y = x + 3$, $y = -x - 3$

Cuts x-axis at $(-3, 0)$

$A_1(3, 6) \quad A_4(3, -6)$

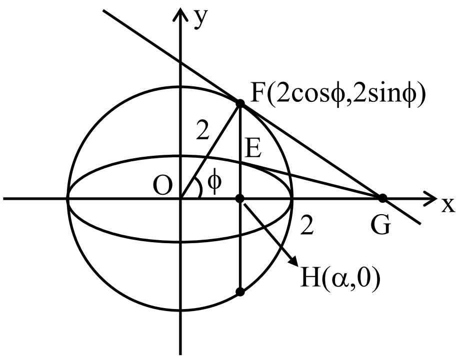
$A_2(-2, 1) \quad A_3(-2, -1)$

$A_1A_4 = 12$, $A_2A_3 = 2$, $MN = 5$

Area $= \frac{1}{2}(12+2)\times 5 = 35$ sq.unit

2. Ans. (C)

Sol.



Let $F(2\cos\phi, 2\sin\phi)$

& $E(2\cos\phi, \sqrt{3}\sin\phi)$

EG : $\frac{x}{2}\cos\phi + \frac{y}{\sqrt{3}}\sin\phi = 1$

$\therefore G\left(\frac{2}{\cos\phi}, 0\right)$ and $\alpha = 2\cos\phi$

ar(ΔFGH) $= \frac{1}{2}$ HG × FH

$= \frac{1}{2}\left(\frac{2}{\cos\phi} - 2\cos\phi\right)\times 2\sin\phi$

$f(\phi) = 2\tan\phi\sin^2\phi$

$\therefore$ (I) $f\left(\frac{\pi}{4}\right) = 1$ (II) $f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$

(III) $f\left(\frac{\pi}{6}\right) = \frac{1}{2\sqrt{3}}$

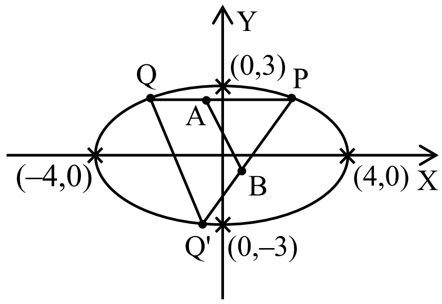
(IV) $f\left(\frac{\pi}{12}\right) = 2\left(2-\sqrt{3}\right)\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2$

$= \left(4-2\sqrt{3}\right)\frac{\left(\sqrt{3}-1\right)^2}{8} = \frac{\left(\sqrt{3}-1\right)^4}{8}$

$\therefore$ (I) $\rightarrow$ (Q) ; (II) $\rightarrow$ (T) ; (III) $\rightarrow$ (S) ; (IV) $\rightarrow$ (P)

3. Ans. (4)

Sol.



A and B be midpoints of segment PQ and PQ' respectively

AB = distance between M(P, Q) and

M(P, Q') $= \frac{1}{2}$.QQ'

Since, Q, Q' must be on E, so,

maximum of QQ' = 8

$\therefore$ Maximum of AB $= \frac{8}{2} = 4$

4. Ans. (A)

Sol. $y^2 = 4\lambda x$, $P(\lambda, 2\lambda)$

Slope of the tangent to the parabola at point P

$\frac{dy}{dx} = \frac{4\lambda}{2y} = \frac{4\lambda}{2x2\lambda} = 1$

Slope of the tangent to the ellipse at P

$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

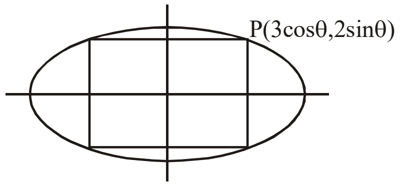
As tangents are perpendicular $y' = -1$

$\Rightarrow \frac{2\lambda}{a^2} - \frac{4\lambda}{b^2} = 0 \Rightarrow \frac{a^2}{b^2} = \frac{1}{2}$

$\Rightarrow e = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

5. Ans. (C, D)

Sol.



Area of $R_1 = 3\sin 2\theta$; for this to be maximum

$\Rightarrow \theta = \frac{\pi}{4} \Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$

Hence for subsequent areas of rectangles R_n to be maximum the coordinates will be in GP with common ratio

$r = \frac{1}{\sqrt{2}} \Rightarrow a_n = \frac{3}{\left(\sqrt{2}\right)^{n-1}}$; $b_n = \frac{3}{\left(\sqrt{2}\right)^{n-1}}$

Eccentricity of all the ellipses will be same

Distance of a focus from the centre in $E_9 = a_9 e_9$

$= \sqrt{a_9^2 - b_9^2} = \frac{\sqrt{5}}{16}$

Length of latus rectum of $E_9 = \frac{2b_9^2}{a_9} = \frac{1}{6}$

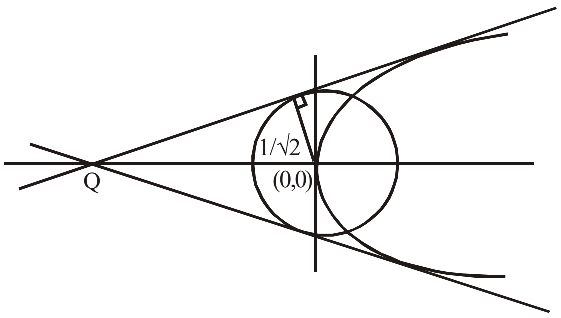
$\because \sum_{n=1}^{\infty} \text{Area of } R_n = 12 + \frac{12}{2} + \frac{12}{4} + ...\infty = 24$

$\Rightarrow \sum_{n=1}^{N} \left(\text{area of } R_n\right) < 24$,

for each positive integer N

6. Ans. (A, C)

Sol.



Let equation of common tangent is

$y = mx + \frac{1}{m}$

$\therefore \left|\frac{0 + 0 + \frac{1}{m}}{\sqrt{1+m^2}}\right| = \frac{1}{\sqrt{2}}$

$\Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$

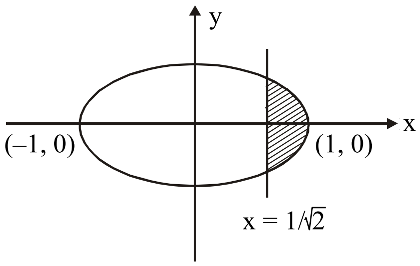
Equation of common tangents are y = x + 1 and y = –x – 1

point Q is (–1, 0)

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$ and $LR = \frac{2b^2}{a} = 1$

(C)



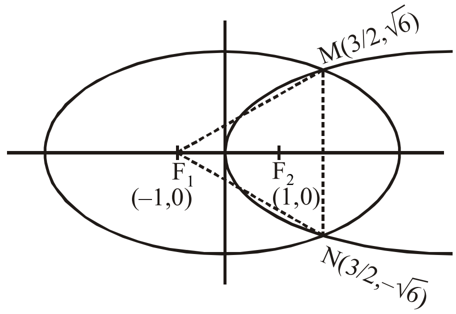
$\text{Area } = 2.\int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}}.\sqrt{1-x^2}\,dx$

$= \sqrt{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_{1/\sqrt{2}}^{1}$

$= \sqrt{2}\left[\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8}\right)\right] = \sqrt{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi - 2}{4\sqrt{2}}$

7. Ans. (A)

Sol.



Orthocentre lies on x-axis

Equation of altitude through M :

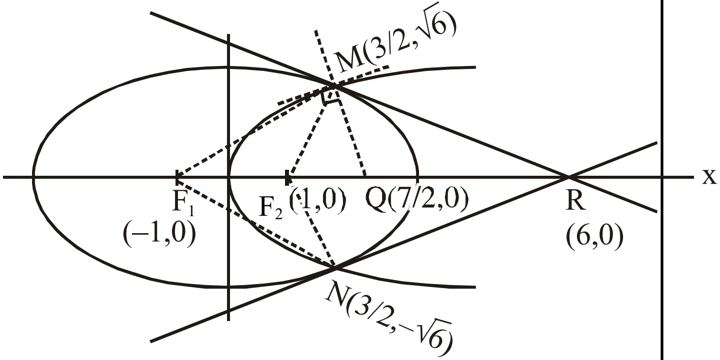
$y - \sqrt{6} = \frac{5}{2\sqrt{6}}\left(x - \frac{3}{2}\right)$

Equation of altitude through F_1 : y = 0

solving, we get orthocentre $\left(-\frac{9}{10}, 0\right)$

8. Ans. (C)

Sol.



Normal to parabola at M :

$$y-\sqrt{6}=-\frac{\sqrt{6}}{2.1}\left(x-\frac{3}{2}\right)$$

Solving it with y = 0, we get $Q\equiv\left(\frac{7}{2},0\right)$

Tangent to ellipse at M : $\frac{x.\frac{3}{2}}{9}+\frac{y(\sqrt{6})}{8}=1$

Solving it with y = 0, we get R ≡ (6, 0)

∴ Area of triangle MQR

$$=\frac{1}{2}.\left(6-\frac{7}{2}\right).\sqrt{6}=\frac{5\sqrt{6}}{4}$$

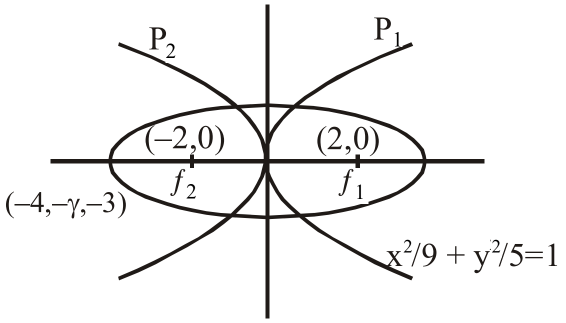
Area of quadrilateral MF_1NF_2

$$=2.\frac{1}{2}.\left(1-(-1)\right).\sqrt{6}=2\sqrt{6}$$

Required ratio = 5 : 8

9. Ans. (4)

Sol.



∴ P_1 is $y^2 = 8x$

P_2 is $y^2 = -16x$

Let equation of tangent at $(2t^2, 4t)$

$$\therefore\ y=m_1x+\frac{2}{m_1}$$

If passes through (–4,0)

$$\therefore\ -4m_1+\frac{2}{m_1}=0$$

$$\therefore\ m_1^2=\frac{1}{2}$$

equation of tangent to P_2

$$y=m_2x+\frac{(-4)}{m_2}$$

It passes through (2, 0), $2m_2-\frac{4}{m_2}=0$

$$\Rightarrow m_2^2=2$$

$$\therefore\ \frac{1}{m_1^2}+m_2^2=4$$

10. Ans. (A, B)

Sol. Let $E_1:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b)

& $E_2:\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ (c < d)

& $S : x^2 + (y-1)^2 = 2$

& tangent to E_1, E_2 & S is x + y = 3

Now, point of contact of S & tangent is (x_1, y_1)

Let x = X & y – 1 = Y

∴ $X^2 + Y^2 = 2$

& X + Y = 2

Let (X_1, Y_1) be point of contact.

∴ $XX_1 + YY_1 = 2$

∴ $X_1 = 1$ & $Y_1 = 1$

∴ $x_1 = 1$ & $y_1 = 2$

Now, parametric equation of x + y = 3

is $\frac{x-1}{-\frac{1}{\sqrt{2}}}=\frac{y-2}{\frac{1}{\sqrt{2}}}=\pm\frac{2\sqrt{2}}{3}\Rightarrow x=\frac{5}{3},\ y=\frac{4}{3}$

$\Rightarrow\quad x=\frac{1}{3}$ & $y=\frac{8}{3}$

∴ P ≡ (1, 2), $Q\equiv\left(\frac{5}{3},\frac{4}{3}\right)$ & $R\equiv\left(\frac{1}{3},\frac{8}{3}\right)$

Now, equation tangent at Q on ellipse E_1

$\frac{x.5}{3a^2}+\frac{y.4}{3b^2}=1$ Comparing it with x + y = 3

∴ $a^2 = 5$ & $b^2 = 4$ Now, $e_1^2=1-\frac{4}{5}=\frac{1}{5}$

Similarly, $e_2^2=\frac{7}{8}$

∴ $e_1^2e_2^2=\frac{7}{40}\quad\Rightarrow\quad e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}}$

$e_1^2+e_2^2=\frac{1}{5}+\frac{7}{8}=\frac{43}{40}$; $|e_1^2-e_2^2|=\left|\frac{1}{5}-\frac{7}{8}\right|=\frac{27}{40}$