DIFFERENTIAL EQUATION

1. Let $f:[(1,\infty) \to \mathbb{R}$ be a differentiable function such that $f(1) = \dfrac{1}{3}$ and $3\displaystyle\int_1^x f(t)\,dt = xf(x) - \dfrac{x^3}{3}, x \in [1,\infty)$.

   Let e denote the base of the natural logarithm. Then the value of $f(e)$ is   **[JEE(Advanced) 2023]**

   (A) $\dfrac{e^2 + 4}{3}$ (B) $\dfrac{\log_e 4 + e}{3}$

   (C) $\dfrac{4e^2}{3}$ (D) $\dfrac{e^2 - 4}{3}$

2. For $x \in \mathbb{R}$, let $y(x)$ be a solution of the differential equation   **[JEE(Advanced) 2023]**

   $(x^2 - 5)\dfrac{dy}{dx} - 2xy = -2x(x^2 - 5)^2$ such that $y(2) = 7$.

   Then the maximum value of the function $y(x)$ is

3. If $y(x)$ is the solution of the differential equation

   $$x\,dy - (y^2 - 4y)dx = 0 \text{ for } x > 0, y(1) = 2,$$

   and the slope of the curve $y = y(x)$ is never zero, then the value of $10y\left(\sqrt{2}\right)$ is _____.

   [JEE(Advanced) 2022]

4. For $x \in \mathbb{R}$, let the function $y(x)$ be the solution of the differential equation

   $$\dfrac{dy}{dx} + 12y = \cos\left(\dfrac{\pi}{12}x\right), \; y(0) = 0.$$

   Then, which of the following statements is/are TRUE?   **[JEE(Advanced) 2022]**

   (A) $y(x)$ is an increasing function

   (B) $y(x)$ is a decreasing function

   (C) There exists a real number β such that the line $y = \beta$ intersects the curve $y = y(x)$ at infinitely many points

   (D) $y(x)$ is a periodic function

5. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation

   $$\dfrac{dy}{dx} + \alpha y = xe^{\beta x}, \; y(1) = 1$$

   Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

   [JEE(Advanced) 2021]

   (A) $f(x) = \dfrac{x^2}{2}e^{-x} + \left(e - \dfrac{1}{2}\right)e^{-x}$ (B) $f(x) = -\dfrac{x^2}{2}e^{-x} + \left(e + \dfrac{1}{2}\right)e^{-x}$

   (C) $f(x) = \dfrac{e^x}{2}\left(x - \dfrac{1}{2}\right) + \left(e - \dfrac{e^2}{4}\right)e^{-x}$ (D) $f(x) = \dfrac{e^x}{2}\left(\dfrac{1}{2} - x\right) + \left(e + \dfrac{e^2}{4}\right)e^{-x}$

6. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $y = \log_e\left(\dfrac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(B) $xy' - \sqrt{1-x^2} = 0$

(C) $y = -\log_e\left(\dfrac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(D) $xy' + \sqrt{1-x^2} = 0$

7. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions.

If $f'(x) = (e^{(f(x) - g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018]**

(A) $f(2) < 1 - \log_e 2$

(B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$

(D) $g(1) < 1 - \log_e 2$

8. Let $f : (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x \text{ for all } x \in (0, \pi).$$

If $f\left(\dfrac{\pi}{6}\right) = -\dfrac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) $f\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{4\sqrt{2}}$

(B) $f(x) < \dfrac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\dfrac{\pi}{2}\right) + f\left(\dfrac{\pi}{2}\right) = 0$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = (2 + 5y)(5y - 2),$$

then the value of $\lim_{x \to -\infty} f(x)$ is _____ . **[JEE(Advanced) 2018]**

10. If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx, \quad x > 0$$

and $y(0) = \sqrt{7}$, then $y(256) =$ **[JEE(Advanced) 2017]**

(A) 80   (B) 3   (C) 16   (D) 9

11. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then

[JEE(Advanced) 2017]

(A) $f(x) > e^{2x}$ in $(0, \infty)$

(B) $f(x)$ is decreasing in $(0, \infty)$

(C) $f(x)$ is increasing in $(0, \infty)$

(D) $f'(x) < e^{2x}$ in $(0, \infty)$

12. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4) \dfrac{dy}{dx} - y^2 = 0, x > 0$, passes through the

point (1,3). The the solution curve-   **[JEE(Advanced) 2016]**

(A) intersects y = x + 2 exactly at one point

(B) intersects y = x + 2 exactly at two points

(C) intersects y = $(x + 2)^2$

(D) does NOT intersect y = $(x + 3)^2$

13. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \dfrac{f(x)}{x}$ for all x $\in (0,\infty)$ and $f(1) \ne 1$.

Then   **[JEE(Advanced) 2016]**

(A) $\lim\limits_{x \to 0^+} f'\left(\dfrac{1}{x}\right) = 1$

(B) $\lim\limits_{x \to 0^+} x f\left(\dfrac{1}{x}\right) = 2$

(C) $\lim\limits_{x \to 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \le 2$ for all x $\in (0,2)$

14. Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If y(0) = 2, then which of the

following statements is(are) true ?   **[JEE(Advanced) 2015]**

(A) y(–4) = 0

(B) y(–2) = 0

(C) y(x) has a critical point in the interval (–1,0)

(D) y(x) has no critical point in the interval (–1,0)

15. Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is

represented by the differential equation Py'' + Qy' + 1 = 0, where P,Q are functions of x,y and y'

(here $y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}$), then which of the following statements is (are) true?   **[JEE(Advanced) 2015]**

(A) P = y + x

(B) P = y – x

(C) P + Q = 1 – x + y + y' + $(y')^2$

(D) P – Q = x + y – y' – $(y')^2$

16. The function y = $f(x)$ is the solution of the differential equation $\dfrac{dy}{dx} + \dfrac{xy}{x^2 - 1} = \dfrac{x^4 + 2x}{\sqrt{1 - x^2}}$ in (–1, 1)

satisfying $f(0) = 0$. Then $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is   **[JEE(Advanced) 2014]**

(A) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$

(B) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$

(C) $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{4}$

(D) $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{2}$

SOLUTIONS

1.   **Ans. (C)**

Sol.   Diff. wr.t 'x'

$$3f(x) = f(x) + xf'(x) - x^2$$

$$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$$

$$IF = e^{-2\ell nx} = \frac{1}{x^2}$$

$$y\left(\frac{1}{x^2}\right) = \int x \cdot \frac{1}{x^2} dx$$

$$y = x^2 \ell nx + cx^2$$

$$\therefore \ y(1) = \frac{1}{3} \Rightarrow c = \frac{1}{3}$$

$$y(e) = \frac{4e^2}{3}$$

2.   **Ans. (16)**

Sol. $\dfrac{dy}{dx} - \dfrac{2x}{x^2 - 5}y = -2x(x^2 - 5)$

$$IF = e^{-\int \frac{2x}{x^2-5}dx} = \frac{1}{(x^2 - 5)}$$

$$y \cdot \frac{1}{x^2 - 5} = \int -2x \cdot dx + c$$

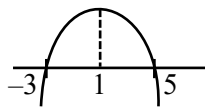$$\Rightarrow \frac{y}{x^2 - 5} = -x^2 + c$$

$$x = 2, y = 7$$

$$\frac{7}{-1} = -4 + c \ \Rightarrow c = -3$$

$$y = -(x^2 - 5)(x^2 + 3)$$

put $x^2 = t \geq 0$

$$y = -(t - 5)(t + 3)$$

$y_{max} = 16$ when $x^2 = 1$

$y_{max} = 16$

3.   **Ans. (8)**

Sol. $xdy - (y^2 - 4y)dx = 0, x > 0$

$$\int \frac{dy}{y^2 - 4y} = \int \frac{dx}{x}$$

$$\int \left(\frac{1}{y-4} - \frac{1}{y}\right)dy = 4 \int \frac{dx}{x}$$

$$\log_e |y - 4| - \log_e |y| = 4 \log_e x + \log_e c$$

$$\frac{|y-4|}{|y|} = cx^4 \xrightarrow{(1,2)} c = 1$$

$$|y - 4| = |y| x^4$$

C-1   and   C-2

$y - 4 = yx^4$ $y - 4 = -yx^4$

$y = \dfrac{4}{1 - x^4}$ $y = \dfrac{4}{1 + x^4}$

$y(1) = ND$ (rejected) $y(1) = 2$

$$y(\sqrt{2}) = \frac{4}{5} \Rightarrow 10y(\sqrt{2}) = 8$$

4.   **Ans. (C)**

Sol. $\dfrac{dy}{dx} + 12y = \cos\left(\dfrac{\pi}{12}x\right)$

Linear D.E.

I.F. $= e^{\int 12 \cdot dx} = e^{12x}$

Solution of DE

$$y \cdot e^{12x} = \int e^{12x} \cdot \cos\left(\frac{\pi}{12}x\right)dx$$

$$y \cdot e^{12x} = \frac{e^{12x}}{(12)^2 + \left(\frac{\pi}{12}\right)^2}\left(12\cos\frac{\pi}{12}x + \frac{\pi}{12}\sin\frac{\pi}{12}x\right) + C$$

$$\Rightarrow y = \frac{(12)}{(12)^4 + \pi^2}\left((12)^2 \cos\left(\frac{\pi x}{12}\right) + \pi \sin\left(\frac{\pi x}{12}\right)\right) + \frac{C}{e^{12x}}$$

$$y = \frac{(12)}{(12)^4 + \pi^2}\left((12)^2 \cos\left(\frac{\pi x}{12}\right) + \pi \sin\left(\frac{\pi x}{12}\right)\right) + \frac{C}{e^{12x}}$$

Given $y(0) = 0$

$$\Rightarrow 0 = \frac{12}{12^4 + \pi^2}(12^2 + 0) + C$$

$$\Rightarrow C = \frac{-12^3}{12^4 + \pi^2}$$

$$\therefore \ y = \frac{12}{12^4 + \pi^2}\left[(12)^2 \cos\left(\frac{\pi x}{12}\right) + \pi \sin\left(\frac{\pi x}{12}\right) - 12^2 \cdot e^{-12x}\right]$$

Now,

$$\frac{dy}{dx} = \frac{12}{12^4 + \pi^2}\left[\underbrace{-12\pi\sin\left(\frac{\pi x}{12}\right) + \frac{\pi^2}{12}\cos\left(\frac{\pi x}{12}\right) + 12^3 e^{-12x}}_{\text{min. value}}\right]$$

$$\left(-\sqrt{144\pi^2 + \frac{\pi^4}{144}} = -12\pi\sqrt{1 + \frac{\pi^2}{12^4}}\right)$$

$\Rightarrow \dfrac{dy}{dx} > 0 \ \forall \ x \le 0$ & may be negative/positive

for $x > 0$

So, $f(x)$ is neither increasing nor decreasing

For some $\beta \in R$, $y = \beta$ intersects $y = f(x)$ at infinitely many points

So, option (C) is correct

5.   **Ans. (A, C)**

Sol.   Integrating factor $= e^{\alpha x}$

So $ye^{\alpha x} = \int xe^{(\alpha+\beta)x}dx$

Case-I

If $\alpha + \beta = 0$ $ye^{\alpha x} = \dfrac{x^2}{2} + c$

It passes through $(1, 1) \Rightarrow C = e^\alpha - \dfrac{1}{2}$

So $ye^{\alpha x} = \dfrac{x^2 - 1}{2} + e^\alpha$

for $\alpha = 1$

$$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x} \rightarrow (A)$$

Case-II

If $\alpha + \beta \ne 0$

$$ye^{\alpha x} = \int \frac{x \cdot e^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{1}{\alpha+\beta}e^{(\alpha+\beta)x}\, dx$$

$$\Rightarrow ye^{\alpha x} = \frac{xe^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + c$$

$$\Rightarrow \text{So } c = e^\alpha - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)^2}\left((\alpha+\beta)x - 1\right) + e^{-\alpha x}$$

$$\left(e^x - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}\right)$$

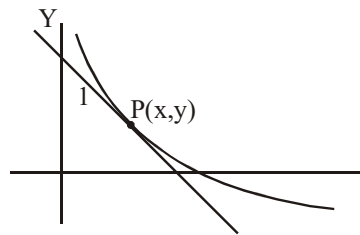If $\alpha = \beta = 1$

$$y = \frac{e^x}{4}(2x-1) + e^{-x}\left(e - \frac{e^2}{2} + \frac{e^2}{4}\right)$$

$$y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + e^{-x}\left(e - \frac{e^2}{4}\right) \rightarrow (c)$$

6.   **Ans. (A, D)**

Sol.

$$Y - y = y'(X - x)$$

So, $Y_P = (0, y - xy')$

So, $x^2 + (xy')^2 = 1 \Rightarrow \dfrac{dy}{dx} = -\sqrt{\dfrac{1-x^2}{x^2}}$

$\left[\dfrac{dy}{dx}\right.$ can not be positive i.e. $f(x)$ can not be increasing in first quadrant, for $x \in (0, 1)]$

Hence, $\displaystyle\int dy = -\int \frac{\sqrt{1-x^2}}{x}\, dx$

$$\Rightarrow y = -\int \frac{\cos^2\theta\, d\theta}{\sin\theta} \ ; \text{ put } x = \sin\theta$$

$$\Rightarrow y = -\int \operatorname{cosec}\theta\, d\theta + \int \sin\theta\, d\theta$$

$$\Rightarrow y = \ln(\operatorname{cosec}\theta + \cot\theta) - \cos\theta + C$$

$$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} + C$$

$$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$$

(as $y(1) = 0$)

7.   **Ans. (B, C)**

Sol. $f'(x) = e^{(f(x) - g(x))} \, g'(x) \; \forall \; x \in \mathbb{R}$

$\Rightarrow \quad e^{-f(x)} \cdot f'(x) - e^{-g(x)} g'(x) = 0$

$\Rightarrow \quad \int \left(e^{-f(x)} f'(x) - e^{-g(x)} \cdot g'(x) \right) dx = C$

$\Rightarrow \quad -e^{-f(x)} + e^{-g(x)} = C$

$\Rightarrow \quad -e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)}$

$\Rightarrow \quad -\dfrac{1}{e} + e^{-g(1)} = -e^{-f(2)} + \dfrac{1}{e}$

$\Rightarrow \quad e^{-f(2)} + e^{-g(1)} = \dfrac{2}{e}$

$\therefore \quad e^{-f(2)} < \dfrac{2}{e}$ and $e^{-g(1)} < \dfrac{2}{e}$

$\Rightarrow \quad -f(2) < \ln 2 - 1$ and $-g(1) < \ln 2 - 1$

$\Rightarrow \quad f(2) > 1 - \ln 2$ and $g(1) > 1 - \ln 2$

8.   **Ans. (B, C, D)**

Sol. $\displaystyle \lim_{t \to x} \dfrac{f(x) \sin t - f(t) \sin x}{t - x} = \sin^2 x$

by using L'Hopital

$\displaystyle \lim_{t \to x} \dfrac{f(x) \cos t - f'(t) \sin x}{1} = \sin^2 x$

$\Rightarrow f(x) \cos x - f'(x) \sin x = \sin^2 x$

$\Rightarrow -\left(\dfrac{f'(x) \sin x - f(x) \cos x}{\sin^2 x} \right) = 1$

$\Rightarrow -d\left(\dfrac{f(x)}{\sin x} \right) = 1$

$\Rightarrow \dfrac{f(x)}{\sin x} = -x + c$

Put $x = \dfrac{\pi}{6}$ & $f\left(\dfrac{\pi}{6} \right) = -\dfrac{\pi}{12}$

$\therefore \; c = 0 \Rightarrow f(x) = -x \sin x$

(A) $f\left(\dfrac{\pi}{4} \right) = \dfrac{-\pi}{4} \dfrac{1}{\sqrt{2}}$
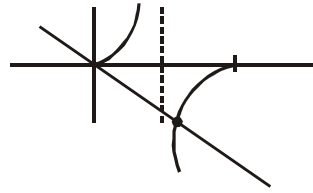
(B) $f(x) = -x \sin x$

as $\sin x > x - \dfrac{x^3}{6}$, $-x \sin x < -x^2 + \dfrac{x^4}{6}$

$\therefore \; f(x) < -x^2 + \dfrac{x^4}{6} \; \forall \; x \in (0, \pi)$

(C) $f'(x) = -\sin x - x \cos x$

$f'(x) = 0 \Rightarrow \tan x = -x$

$\Rightarrow$ there exist $\alpha \in (0, \pi)$ for which $f'(\alpha) = 0$

(D) $f''(x) = -2\cos x + x \sin x$

$f''\left(\dfrac{\pi}{2} \right) = \dfrac{\pi}{2}$, $f\left(\dfrac{\pi}{2} \right) = -\dfrac{\pi}{2}$

$f''\left(\dfrac{\pi}{2} \right) + f\left(\dfrac{\pi}{2} \right) = 0$

9.   **Ans. (0.4)**

Sol. $\dfrac{dy}{dx} = 25y^2 - 4$

So, $\dfrac{dy}{25y^2 - 4} = dx$

Integrating, $\dfrac{1}{25} \times \dfrac{1}{2 \times \frac{2}{5}} \ln \left| \dfrac{y - \frac{2}{5}}{y + \frac{2}{5}} \right| = x + c$

$\Rightarrow \ln \left| \dfrac{5y - 2}{5y + 2} \right| = 20(x + c)$

Now, $c = 0$ as $f(0) = 0$

Hence $\left| \dfrac{5y - 2}{5y + 2} \right| = e^{(20x)}$

$\displaystyle \lim_{x \to -\infty} \left| \dfrac{5f(x) - 2}{5f(x) + 2} \right| = \lim_{x \to -\infty} e^{(20x)}$

Now, RHS $= 0 \; \Rightarrow \; \displaystyle \lim_{x \to -\infty} (5f(x) - 2) = 0$

$\Rightarrow \; \displaystyle \lim_{x \to -\infty} f(x) = \dfrac{2}{5}$

10.   **Ans. (B)**

Sol. $y = \dfrac{1}{8} \displaystyle \int \dfrac{dx}{\sqrt{4 + \sqrt{9 + x} \cdot \sqrt{x} \cdot \sqrt{9 + \sqrt{x}}}}$

put $\sqrt{9 + \sqrt{x}} = t \Rightarrow \dfrac{dx}{\sqrt{x} \cdot \sqrt{9 + \sqrt{x}}} = 4dt$

$\therefore \ y = \dfrac{4}{8} \displaystyle\int \dfrac{dt}{\sqrt{4+t}}$

$\Rightarrow \ y = \sqrt{4+t} + C$

$\Rightarrow \ y(x) = \sqrt{4 + \sqrt{9 + \sqrt{x}}} + C$

at $x = 0 : y(0) = \sqrt{7} \quad \Rightarrow \quad C = 0$

$\therefore \ y(x) = \sqrt{4 + \sqrt{9 + \sqrt{x}}}$

$\Rightarrow \quad y(256) = 3$

11.   **Ans. (A, C)**

Sol.   Given that,

$\qquad f'(x) > 2f(x) \ \forall \ x \in R$

$\Rightarrow \quad f'(x) - 2f(x) > 0 \quad \forall \ x \in R$

$\therefore \quad e^{-2x}(f'(x) - 2f(x)) > 0 \ \forall \ x \in R$

$\Rightarrow \quad \dfrac{d}{dx}\left(e^{-2x}f(x)\right) > 0 \ \forall \ x \in R$

Let $g(x) = e^{-2x}f(x)$

Now, $g'(x) > 0 \ \forall \ x \in R$

$\Rightarrow \quad g(x)$ is strictly increasing $\forall \ x \in R$

Also, $g(0) = 1$

$\therefore \quad \forall \ x > 0$

$\Rightarrow \quad g(x) > g(0) = 1$

$\therefore \quad e^{-2x}.f(x) > 1 \ \forall \ x \in (0, \infty) \Rightarrow f(x) > e^{2x} \ \forall$

$\quad x \in (0,\infty)$

$\therefore \quad$ option (A) is correct

As, $f'(x) > 2\, f(x) > 2e^{2x} > 2 \ \forall \ x \in (0, \infty)$

$\Rightarrow \quad f(x)$ is strictly increasing on $x \in (0, \infty)$

$\Rightarrow \quad$ option (C) is correct

As, we have proved above that

$\qquad f'(x) > 2.e^{2x} \ \forall \ x \in (0, \infty)$

$\Rightarrow \quad$ option (D) is incorrect

$\therefore \quad$ options (A) and (C) are correct

12.   **Ans. (A, D)**

Sol. $(x^2 + xy + 4x + 2y + 4)\dfrac{dy}{dx} - y^2 = 0$

$\qquad ((x+2)^2 + y(x+2))\dfrac{dy}{dx} = y^2$

Let $x + 2 = X, \ y = Y$

$(X)(X + Y)\dfrac{dY}{dX} = Y^2$

$-X^2 dY = XY dY - Y^2 dX$

$-X^2 dY = Y(X dY - Y dX)$

$-\dfrac{dY}{Y} = \dfrac{X dY - Y dX}{X^2}$

$-\ell n\,|Y| = \left(\dfrac{Y}{X}\right) + C$

$-\ell n\,|y| = \dfrac{y}{x+2} + C$

$\because \quad$ it is passing through (1, 3)

$\qquad -\ell n 3 = 1 + C$

$\qquad C = -1 - \ell n 3$

$\therefore$ curve $\dfrac{y}{x+2} + \ell n\,|y| - 1 - \ell n 3 = 0, x > 0 \ ....(i)$

put $y = x + 2$ in equation (i)

then $\dfrac{x+2}{x+2} + \ell n\,|x+2| - 1 - \ell n 3 = 0$

$x = 1, -5(\text{reject})$

$\therefore$ curve intersect $y = x + 2$ at point (1, 3)

for option (C), put $y = (x + 2)^2$, we will get

$x + 2 + 2\ell n(x + 2) = 1 + \ell n 3$

Clearly left hand side is an increasing function

Hence, it is always greater than $2 + 2\ell n 2$

therefore no solution

for option (C) put $y = (x + 3)^2$ in equation (i)

$\dfrac{(x+3)^2}{x+2} + \ell n(x+3)^2 - 1 - \ell n 3 = 0$

$\dfrac{(x+3)^2}{x+2} + \ell n \dfrac{(x+3)^2}{3} - 1 = 0$

$\because \quad x > 0 \ \Rightarrow x + 3 > x + 2$ and $x + 3 > 3$

So $\dfrac{(x+3)^2}{x+2} + \ell n \dfrac{(x+3)^2}{3} > 1$

$\therefore \quad \dfrac{(x+3)^2}{x+2} + \ell n \dfrac{(x+3)^2}{3} - 1 = 0$

has no solution

$\Rightarrow \quad$ curve $y = (x + 3)^2$ does not intersect

13.   **Ans. (A)**

Sol.   Let $y = f(x)$

$$\frac{dy}{dx} + \frac{y}{x} = 2 \quad \text{(linear differential equation)}$$

$$\therefore \; y.e^{\int \frac{dx}{x}} = 2\int e^{\int \frac{dx}{x}} = 2\int e^{\int \frac{dx}{x}} dx + c$$

$$\Rightarrow yx = 2\int x\,dx + c$$

$$\therefore \; yx = x^2 + c$$

$$\Rightarrow f(x) = x + \frac{c}{x} \; ; \; \text{As } f(1) \neq 1 \Rightarrow c \neq 0$$

$$\Rightarrow \; f'(x) = 1 - \frac{c}{x^2}, c \neq 0$$

(A) $\displaystyle \lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x \to 0^+}\left(1 - cx^2\right) = 1$

(B) $\displaystyle \lim_{x \to 0^+} xf\left(\frac{1}{x}\right) = \lim_{x \to 0^+} x\left(\frac{1}{x} + cx\right) = \lim_{x \to 0^+}\left(1 + cx^2\right) = 1$

(C) $\displaystyle \lim_{x \to 0^+} x^2 f'(x) = \lim_{x \to 0^+} x^2\left(1 - \frac{c}{x^2}\right) = \lim_{x \to 0^+}\left(x^2 - c\right) = -c$

(D) $f(x) = x + \dfrac{c}{x}, c \neq 0$

for $c > 0$

$\therefore \displaystyle \lim_{x \to 0^+} f(x) = \infty$

$\Rightarrow$ function is not bounded in $(0, 2)$

14.   **Ans. (A, C)**

Sol. $y' + e^x y' + y e^x = 1$

$\Rightarrow \quad dy + d(e^x y) = dx$

$\Rightarrow \quad y + e^x y = x + c$

$\because \quad y(0) = 2 \quad \Rightarrow \quad c = 4$

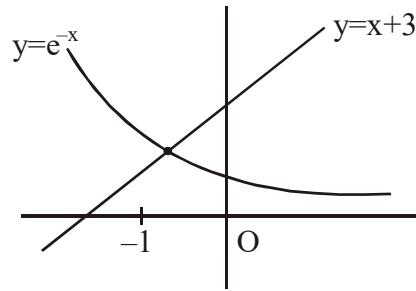$\Rightarrow \quad y = \dfrac{x+4}{1+e^x}$

$\therefore \quad y(-4) = 0$

for critical point given

$$\frac{dy}{dx} = \frac{1 - y e^x}{1 + e^x} = \frac{1 - \left(\dfrac{x+4}{1+e^x}\right)e^x}{1+e^x} = \frac{1-(x+3)e^x}{(1+e^x)^2}$$

$\because \quad \dfrac{dy}{dx} = 0 \quad \Rightarrow \quad e^x(x+3) - 1 = 0$

$\Rightarrow \quad x + 3 = e^{-x}$

$y(x)$ has a critical point in the interval $(-1, 0)$

15.   **Ans. (B, C)**

Sol.   Let Circle

$$x^2 + y^2 - 2ax - 2ay + c = 0$$

On differentiation

$$2x + 2yy' - 2a - 2ay' = 0$$

$$\Rightarrow \quad x + yy' - a(1 + y') = 0$$

$$\Rightarrow \quad a = \frac{x + yy'}{1 + y'}$$

again differentiation

$$\frac{(1 + (y')^2 + yy'')(1 + y') - (x + yy')(y'')}{(1 + y')^2} = 0$$

$$\Rightarrow \quad 1 + y'((y')^2 + y' + 1) + y''(y - x) = 0$$

$$\therefore \quad P = y - x$$

$$Q = 1 + y' + (y')^2$$

16.   **Ans. (B)**

Sol. $\left(\dfrac{dy}{dx} - \dfrac{xy}{(1-x^2)}\right)\sqrt{1-x^2} = x^4 + 2x$

$$\Rightarrow \quad \sqrt{1-x^2}\,dy + d\left(\sqrt{1-x^2}\right)y = (x^4 + 2x)dx$$

$$\Rightarrow \quad y\sqrt{1-x^2} = \frac{x^5}{5} + x^2 + c$$

by $(0, 0)$ $c = 0$

$$y = \frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \left(\frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}\right)dx$$

$$= 2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx \qquad \text{(put } x = \sin\theta\text{)}$$

$$= 2\int_0^{\pi/3} \sin^2\theta\,d\theta = \int_0^{\pi/3}(1 - \cos 2\theta)d\theta$$

$$= \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$