COMPLEX NUMBER

1. Let $A=\left\{\frac{1967+1686i\sin\theta}{7-3i\cos\theta}:\theta\in\mathbb{R}\right\}$. If A contains exactly one positive integer n, then the value of n is

[JEE(Advanced) 2023]

2. Let z be complex number satisfying $|z|^3+2z^2+4\bar{z}-8=0$, where $\bar{z}$ denotes the complex conjugate of z. Let the imaginary part of z be nonzero. **[JEE(Advanced) 2023]**

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	List-II
(P) $\lvert z\rvert^2$ is equal to	(1) 12
(Q) $\lvert z-\bar{z}\rvert^2$ is equal to	(2) 4
(R) $\lvert z\rvert^2+\lvert z+\bar{z}\rvert^2$ is equal to	(3) 8
(S) $\lvert z+1\rvert^2$ is equal to	(4) 10
	(5) 7

The correct option is :

(A) (P) → (1) (Q) → (3) (R) → (5) (S) → (4)

(B) (P) → (2) (Q) → (1) (R) → (3) (S) → (5)

(C) (P) → (2) (Q) → (4) (R) → (5) (S) → (1)

(D) (P) → (2) (Q) → (3) (R) → (5) (S) → (4)

3. Let A_1, A_2, A_3, ..., A_8 be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle and let PA_i denote the distance between the points P and A_i for i = 1,2,....,8. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \cdots PA_8$, is

[JEE(Advanced) 2023]

4. Let z be a complex number with non-zero imaginary part. If

$$\frac{2+3z+4z^2}{2-3z+4z^2}$$

is a real number, then the value of $|z|^2$ is _______. **[JEE(Advanced) 2022]**

5. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $i=\sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation

$$\bar{z}-z^2=i(\bar{z}+z^2)$$

is _______. **[JEE(Advanced) 2022]**

6. Let $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of

$$(\bar{z})^2+\frac{1}{z^2}$$

are integers, then which of the following is/are possible value(s) of |z| ? **[JEE(Advanced) 2022]**

(A) $\left(\frac{43+3\sqrt{205}}{2}\right)^{\frac{1}{4}}$ (B) $\left(\frac{7+\sqrt{33}}{4}\right)^{\frac{1}{4}}$

(C) $\left(\frac{9+\sqrt{65}}{4}\right)^{\frac{1}{4}}$ (D) $\left(\frac{7+\sqrt{13}}{6}\right)^{\frac{1}{4}}$

7. Let $\theta_1, \theta_2, ..., \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + ... + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for k = 2, 3, ..., 10, where $i = \sqrt{-1}$. Consider the statements P and Q given below : **[JEE(Advanced) 2021]**

$P : |z_2 - z_1| + |z_3 - z_2| + ... + |z_{10} - z_9| + |z_1 - z_{10}| \le 2\pi$

$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \le 4\pi$

Then,

(A) P is **TRUE** and Q is **FALSE**
(B) Q is **TRUE** and P is **FALSE**
(C) both P and Q are **TRUE**
(D) both P and Q are **FALSE**

8. For any complex number w = c + id, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers z = x + iy satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x,y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0.$$

Then which of the following statements is (are) **TRUE** ? **[JEE(Advanced) 2021]**

(A) $\alpha = -1$
(B) $\alpha\beta = 4$
(C) $\alpha\beta = -4$
(D) $\beta = 4$

9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) $\left|z + \frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$
(B) $|z| \le 2$ for all $z \in S$
(C) $\left|z + \frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$
(D) The set S has exactly four elements

10. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $Re(z_1) > 0$ and $Re(z_2) < 0$, is _____. **[JEE(Advanced) 2020]**

11. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \overline{z}_0}{z_0 - \overline{z}_0 + 2i}$ is

[JEE(Advanced) 2019]

(A) $\frac{\pi}{4}$
(B) $-\frac{\pi}{2}$
(C) $\frac{3\pi}{4}$
(D) $\frac{\pi}{2}$

12. Let $\omega \ne 1$ be a cube root of unity. Then the minimum of the set

$$\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$$

equals _____. **[JEE(Advanced) 2019]**

13. For a non-zero complex number z, let arg(z) denotes the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement(s) is (are) **FALSE**? **[JEE(Advanced) 2018]**

(A) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(B) The function $f : \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

14. Let s, t, r be the non-zero complex numbers and L be the set of solutions z = x + iy $\left(x, y \in \mathbb{R}, i = \sqrt{-1}\right)$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) If L has exactly one element, then $|s| \ne |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

15. Let a, b, x and y be real numbers such that a – b = 1 and y ≠ 0. If the complex number z = x + iy satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ? **[JEE(Advanced) 2017]**

(A) $-1-\sqrt{1-y^2}$ (B) $1+\sqrt{1+y^2}$ (C) $1-\sqrt{1+y^2}$ (D) $-1+\sqrt{1-y^2}$

16. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s ∈ {1, 2, 3}. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is **[JEE(Advanced) 2016]**

17. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \ne 0$. Suppose $S = \left\{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \ne 0\right\}$, where.

If z = x + iy and z ∈ S, then (x,y) lies on **[JEE(Advanced) 2016]**

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for a > 0, b ≠ 0

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for a < 0, b ≠ 0

(C) the x-axis for a ≠ 0, b = 0

(D) the y-axis for a = 0, b ≠ 0

18. **[JEE(Advanced) 2015]**

	Column-I		**Column-II**
(A)	In $\mathbb{R}^2$, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P)	1
(B)	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2-2, & x<1 \\ bx+a^2, & x\ge 1 \end{cases}$ is differentiable for all $x \in \mathbb{R}$. Then possible value(s) of a is (are)	(Q)	2
(C)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3} + (2+3\omega-3\omega^2)^{4n+3} + (-3+2\omega+3\omega^2)^{4n+3} = 0$ then possible value(s) of n is (are)	(R)	3
(D)	Let the harmonic mean of two positive real number a and b be 4, If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	(S)	4
		(T)	5

19. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i=\sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}\lvert\alpha_{k+1}-\alpha_k\rvert}{\sum_{k=1}^{3}\lvert\alpha_{4k-1}-\alpha_{4k-2}\rvert}$$ is **[JEE(Advanced) 2015]**

20. The quadratic equation p(x) = 0 with real coefficients has purely imaginary roots. Then the equation p(p(x)) = 0 has **[JEE(Advanced) 2014]**

(A) only purely imaginary roots
(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots.

21. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; k = 1, 2, ... 9. **[JEE(Advanced) 2014]**

	List-I		**List-II**
P.	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	1.	True
Q.	There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	2.	False
R.	$\frac{\lvert 1-z_1\rvert\lvert 1-z_2\rvert ... \lvert 1-z_9\rvert}{10}$ equals	3.	1
S.	$1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	4.	2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

SOLUTIONS

1. Ans. (281)

Sol. $A = \dfrac{1967 + 1686 i \sin\theta}{7 - 3i\cos\theta}$

$= \dfrac{281(7 + 6i\sin\theta)}{7 - 3i\cos\theta} \times \dfrac{7 + 3i\cos\theta}{7 + 3i\cos\theta}$

$= \dfrac{281(49 - 18\sin\theta\cos\theta + i(21\cos\theta + 42\sin\theta))}{49 + 9\cos^2\theta}$

for positive integer

Im(A) = 0

$21\cos\theta + 42\sin\theta = 0$

$\tan\theta = \dfrac{-1}{2}$; $\sin 2\theta = \dfrac{-4}{5}$, $\cos^2\theta = \dfrac{4}{5}$

$\text{Re}(A) = \dfrac{281(49 - 9\sin 2\theta)}{49 + 9\cos^2\theta}$

$= \dfrac{281\left(49 - 9 \times \dfrac{-4}{5}\right)}{49 + 9 \times \dfrac{4}{5}}$ = 281 (+ve integer)

2. Ans. (B)

Sol. $\because |z|^3 + 2z^2 + 4\bar{z} - 8 = 0$ …. (1)

Take conjugate both sides

$\Rightarrow |z|^3 + 2\bar{z}^2 + 4z - 8 = 0$ …. (2)

By (1) – (2)

$\Rightarrow 2(z^2 - \bar{z}^2) + 4(\bar{z} - z) = 0$

$\Rightarrow \boxed{z + \bar{z} = 2}$ …. (3)

$\Rightarrow |z + \bar{z}| = 2$ …. (4)

Let z = x + iy

$\therefore \boxed{x = 1}$ $\therefore$ z = 1 + iy

Put in (1)

$\Rightarrow (1 + y^2)^{3/2} + 2(1 - y^2 + 2iy) + 4(1 - iy) - 8 = 0$

$\Rightarrow (1 + y^2)^{3/2} = 2(1 + y^2)$

$\Rightarrow \sqrt{1 + y^2} = 2 = |z|$

Also $\boxed{y = \pm\sqrt{3}}$

$\therefore z = 1 \pm i\sqrt{3}$

$\Rightarrow z - \bar{z} = \pm 2i\sqrt{3}$

$\Rightarrow |z - \bar{z}| = 2\sqrt{3}$

$\Rightarrow |z - \bar{z}|^2 = 12$

Now $z + 1 = 2 + i\sqrt{3}$

$|z + 1|^2 = 4 + 3 = 7$

$\therefore P \to 2$; $Q \to 1$; $R \to 3$; $S \to 5$

$\therefore$ Option (B) is correct.

3. Ans. (512)

Sol. $z^8 - 2^8 = (z - 2)(z - \alpha)(z - \alpha^2)\ldots(z - \alpha^7)$

Put $z = 2e^{i\theta}$

$2^8(e^{i8\theta} - 1) = (2e^{i\theta} - 2)(2e^{i\theta} - \alpha)\ldots..(2e^{i\theta} - \alpha^7)$

Take mod

$2^8|e^{i8\theta} - 1| = PA_1\, PA_2 \ldots PA_8$

$2^8|2\sin 4\theta| = PA_1\, PA_2 \ldots PA_8$

$(PA_1 \cdot PA_2 \ldots PA_8)_{max} = 512$

4. Ans. (0.50)

Sol. Given that

$z \neq \bar{z}$

Let $\alpha = \dfrac{2 + 3z + 4z^2}{2 - 3z + 4z^2} = \dfrac{(2 - 3z + 4z^2) + 6z}{2 - 3z + 4z^2}$

$\therefore \alpha = 1 + \dfrac{6z}{2 - 3z + 4z^2}$

If α is a real number, then

$\alpha = \bar{\alpha}$

$\Rightarrow \dfrac{z}{2 - 3z + 4z^2} = \dfrac{\bar{z}}{2 - 3\bar{z} + 4\bar{z}^2}$

$\therefore 2(z - \bar{z}) = 4z\bar{z}(z - \bar{z})$

$\Rightarrow (z - \bar{z})(2 - 4z\bar{z}) = 0$

As $z \neq \bar{z}$ (Given)

$\Rightarrow z\bar{z} = \dfrac{2}{4} = \dfrac{1}{2}$

$\Rightarrow |z|^2 = 0.50$

5. Ans. (4.00)

Sol. Given ,

$\bar{z} - z^2 = i(\bar{z} + z^2)$

$\Rightarrow (1 - i)\bar{z} = (1 + i)z^2$

$\Rightarrow \dfrac{(1 - i)}{(1 + i)}\bar{z} = z^2$

$\Rightarrow \left(-\dfrac{2i}{2}\right)\bar{z} = z^2$

$\therefore\ z^2 = -i\,\bar{z}$

Let z = x + iy,

$\therefore\ (x^2 - y^2) + i\,(2xy) = -i\,(x - iy)$

so, $x^2 - y^2 + y = 0$ …(1)

and $(2y + 1)x = 0$ …(2)

$\Rightarrow x = 0$ or $y = -\frac{1}{2}$

Case I : When x = 0

$\therefore$ (1) $\Rightarrow y(1 - y) = 0 \Rightarrow y = 0,1$

$\therefore$ (0,0), (0,1)

Case II : When $y = -\frac{1}{2}$

$\therefore$ (1) $\Rightarrow$ $x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 = \frac{3}{4}$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2}$

$\therefore \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\Rightarrow$ Number of distinct 'z' is equal to 4.

6. Ans. (A)

Sol. Let $(\bar{z})^2 + \frac{1}{z^2} = m + in$, $m, n \in \mathbb{Z}$

$(\bar{z})^2 + \frac{\bar{z}^2}{|z|^4} = m + in$

$\Rightarrow (x^2 - y^2)\left(1 + \frac{1}{|z|^4}\right) = m$ …(1)

& $-2xy\left(1 + \frac{1}{|z|^4}\right) = n$ …(2)

Equation $(1)^2 + (2)^2$

$\left(1 + \frac{1}{|z|^4}\right)^2 \left[\left(x^2 + y^2\right)^2\right] = m^2 + n^2$

$\left(1 + \frac{1}{|z|^4}\right)^2 (|z|)^4 = m^2 + n^2$

$\Rightarrow |z|^4 + \frac{1}{|z|^4} + 2 = m^2 + n^2$

Now for option (A)

$|z|^4 = \frac{43 + 3\sqrt{205}}{2}$

$\Rightarrow m^2 + n^2 = 45$

$\Rightarrow m = \pm 6,\ n = \pm 3$

Option (B)

$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{33}}{4} + \frac{7 - \sqrt{33}}{4} + 2$

$= \frac{7}{2} + 2 = \frac{11}{2}$

Option (C)

$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{9 + \sqrt{65}}{4} + \frac{9 - \sqrt{65}}{4} + 2$

$= \frac{9}{2} + 2 = \frac{13}{2}$

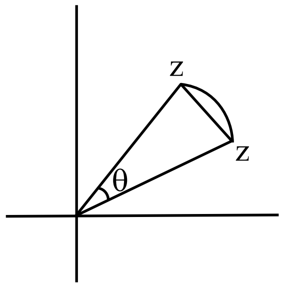
Option (D)

$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{13}}{6} + \frac{7 - \sqrt{13}}{6} + 2$

$= \frac{7}{3} + 2 = \frac{13}{2}$

7. Ans. (C)

Sol.



$|z_1| = |z_2| = \ldots |z_{10}| = 1$

angle $= \frac{\text{arc}}{\text{rad}}$

$\theta_2 = \text{arc}(z_1 z_2) > |z_2 - z_1|$

P : $|z_2 - z_1| + \ldots + |z_1 - z_{10}| \le \theta_1 + \theta_2 + \ldots + \theta_{10}$

$\Rightarrow |z_2 - z_1| + \ldots + |z_1 - z_{10}| \le 2\pi$ P is true

$z_1^2 = e^{i2\theta_1}$, $z_k^2 = z_{k-1}^2 . e^{i2\theta_k}$

Let $2\theta_k = \alpha k$

$z_1^2 = e^{i\alpha_1}$, $z_k^2 = z_{k-1}^2 . e^{i\alpha_k}$

$\alpha_1 + \alpha_2 + \ldots + \alpha_k = 4\pi$

one similar sense

$|z_1^2 - z_2^2| + \ldots |z_1^2 - z_{10}^2| \le 4\pi$

Q is also true

8. Ans. (B, D)

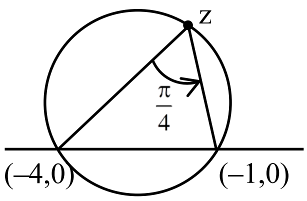
Sol. $\arg\left(\dfrac{z+\alpha}{z+\beta}\right)=\dfrac{\pi}{4}$ implies z is

on arc and $(-\alpha, 0)$ & $(-\beta, 0)$ subtend $\dfrac{\pi}{4}$ on z.

And z lies on $x^2 + y^2 + 5x - 3y + 4 = 0$

So put y = 0;

$x^2 + 5x + 4 = 0 \Rightarrow x = -1$; $x = -4$



Now, $\arg\left(\dfrac{z+\alpha}{z+\beta}\right)=\dfrac{\pi}{4}$

$\Rightarrow z + \alpha = (z + \beta) \cdot r \cdot e^{i\frac{\pi}{4}}$

So, $z + \beta = z + 4 \Rightarrow \beta = 4$ & $z + \alpha = z + 1$

$\Rightarrow \alpha = 1$

9. Ans. (B, C)

Sol. $|z^2 + z + 1| = 1$

$\Rightarrow \left|\left(z+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\right|=1$

$\Rightarrow \left|\left(z+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\right|\le\left|z+\dfrac{1}{2}\right|^2+\dfrac{3}{4}$

$\Rightarrow 1\le\left|z+\dfrac{1}{2}\right|^2+\dfrac{3}{4} \quad \Rightarrow \left|\left(z+\dfrac{1}{2}\right)\right|^2\ge\dfrac{1}{4}$

$\Rightarrow \left|z+\dfrac{1}{2}\right|\ge\dfrac{1}{2}$

also $|(z^2 + z) + 1| = 1 \ge ||z^2 + z| - 1|$

$\Rightarrow |z^2 + z| - 1 \le 1$

$\Rightarrow |z^2 + z| \le 2$

$\Rightarrow ||z^2| - |z|| \le |z^2 + z| \le 2$

$\Rightarrow |r^2 - r| \le 2$

$\Rightarrow r = |z| \le 2$; $\forall z \in S$

Also we can always find root of the equation $z^2 + z + 1 = e^{i\theta}$; $\forall \theta \in R$

Hence set 'S' is infinite

10. Ans. (8)

Sol. Let $z = x + iy$

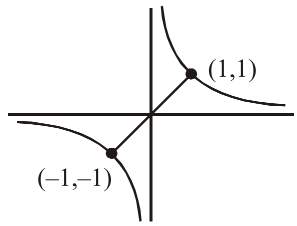
$z^4 - |z|^4 = 4iz^2$

$\Rightarrow z^4-(z\bar{z})^2=4iz^2$

$\Rightarrow z = 0$ or $z^2-(\bar{z})^2=4i$

$\Rightarrow 4ixy = 4i$

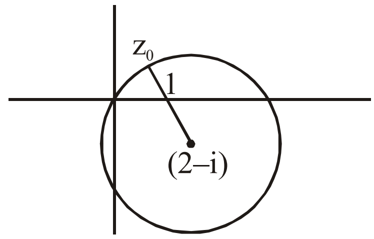
$\Rightarrow xy = 1$



$|z_1 - z_2|^2_{min} = 8$

11. Ans. (B)

Sol.



$\arg\left(\dfrac{4-(z_0+\bar{z}_0)}{(z_0-\bar{z}_0)+2i}\right)$

$=\arg\left(\dfrac{4-2\operatorname{Re}z_0}{2i\operatorname{Im}z_0+2i}\right)=\arg\left(\dfrac{2-\operatorname{Re}z_0}{(1+\operatorname{Im}z_0)i}\right)$

$=\arg\left(-\left(\dfrac{2-\operatorname{Re}z_0}{1+\operatorname{Im}z_0}\right)i\right)$

$= \arg(-ki)$; $k > 0$ (as $\operatorname{Re}z_0 < 2$ & $\operatorname{Im}z_0 > 0$)

$=-\dfrac{\pi}{2}$

12. Ans. (3.00)

Sol. $|a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2)\left(\overline{a+b\omega+c\omega^2}\right)$

$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= a^2 + b^2 + c^2 - ab - bc - ca$

$=\dfrac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

$\ge\dfrac{1+1+4}{2}=3$ (when a = 1, b = 2, c = 3)

13. Ans. (A, B, D)

Sol. (A) $\arg(-1-i) = -\frac{3\pi}{4}$,

(B) $f(t) = \arg(-1 + it)$

$$= \begin{cases} \pi - \tan^{-1}(t), & t \ge 0 \\ -\pi + \tan^{-1}(-t), & t < 0 \end{cases}$$

Discontinuous at t = 0.

(C) $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$

$= \arg z_1 - \arg(z_2) + 2n\pi - \arg(z_1) + \arg(z_2)$
$= 2n\pi$.

(D) $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$

$\Rightarrow \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ is real.

$\Rightarrow$ z, z_1, z_2, z_3 are concyclic.

14. Ans. (A, C, D)

Sol. Given

$sz + t\bar{z} + r = 0$ ....(1)

on taking conjugate $\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0$ ....(2)

from (1) and (2) eliminating $\bar{z}$

$z\left(|s|^2 - |t|^2\right) = \bar{r}t - r\bar{s}$

(A) If |s| ≠ |t| then z has unique value

(B) If |s| = |t| then $\bar{r}t - r\bar{s}$ may or may not be zero so L may be empty set

(C) locus of z is null set or singleton set or a line in all cases it will intersect given circle at most two points.

(D) In this case locus of z is a line so L has infinite elements

15. Ans. (A, D)

Sol. $\text{Im}\left(\frac{az+b}{z+1}\right) = y$ and z = x + iy

$\therefore \text{Im}\left(\frac{a(x+iy)+b}{x+iy+1}\right) = y$

$\Rightarrow \text{Im}\left(\frac{(ax+b+iay)(x+1-iy)}{(x+1)^2+y^2}\right) = y$

$\Rightarrow -y(ax+b) + ay(x+1)$

$= y((x+1)^2 + y^2)$

$\Rightarrow (a-b)y = y((x+1)^2 + y^2)$

$\because$ $y \ne 0$ and $a - b = 1$

$\Rightarrow (x+1)^2 + y^2 = 1$

$\Rightarrow x = -1 \pm \sqrt{1-y^2}$

16. Ans. (1)

Sol. $z = \omega$

$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$, $P^2 = -I$

$\Rightarrow P^2 = \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s}((-1)^r + 1) \\ \omega^{r+2s}((-1)^r + 1) & \omega^{4s} + \omega^{2r} \end{bmatrix} = -I$

$\Rightarrow (-1)^r + 1 = 0 \Rightarrow$ r is odd $\Rightarrow$ r = 1,3

also $\omega^{2r} + \omega^{4s} = -1 \therefore$ $r \ne 3$

by r = 1 $\Rightarrow \omega^2 + \omega^{4s} = -1 \Rightarrow s = 1$

(r, s) = (1, 1)

only 1 pair

17. Ans. (A, C, D)

Sol.. $x + iy = \frac{1}{a+ibt}$

$x + iy = \frac{a - ibt}{a^2 + b^2t^2}$

Let $a \ne 0$ & $b \ne 0$

$x = \frac{a}{a^2+b^2t^2}$ ....(1)

$y = \frac{-bt}{a^2+b^2t^2}$ ....(2)

$\frac{y}{x} = \frac{-bt}{a} \Rightarrow t = -\frac{ay}{bx}$

put in (1)

$x\left\{a^2 + b^2.\frac{a^2y^2}{b^2x^2}\right\} = a$

$a^2(x^2 + y^2) = ax$

$x^2 + y^2 - \frac{1}{a}x = 0$

$\left(x - \frac{1}{2a}\right)^2 + y^2 = \frac{1}{4a^2}$

$\Rightarrow$ option (A) is correct

for $a \neq 0, b = 0$

$$x + iy = \frac{1}{a}$$

$$x = \frac{1}{a},\ y = 0 \ \Rightarrow \text{ z lies on x-axis}$$

$\Rightarrow$ option (C) is correct

for $a = 0, b \neq 0$

$$x + iy = \frac{1}{ibt}$$

$$y = -\frac{1}{bt}i,\ x = 0$$

$\Rightarrow$ z lies on y-axis.

$\Rightarrow$ option (D) is correct

18. Ans. (A) $\to$ (P,Q); (B) $\to$ (P,Q);
(C) $\to$ (P,Q,S,T); (D) $\to$ (Q,T)

Sol. (A) $\left|\frac{\alpha\sqrt{3}+\beta}{2}\right| = \sqrt{3} \Rightarrow \alpha\sqrt{3} + \frac{\alpha-2}{\sqrt{3}} = \pm 2\sqrt{3}$

$$\Rightarrow \alpha\left(\frac{4}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} \pm 2\sqrt{3}$$

$$\alpha = 2, -1 \Rightarrow |\alpha| = 1, 2$$

(B) By continuity $\quad -3a - 2 = b + a^2$

By differentiability $-6a = b$

$a^2 - 3a + 2 = 0 \Rightarrow a = 1,2$

(C) $\left(\left(-3+2\omega+3\omega^2\right)\omega\right)^{4n+3} +$

$\left(\left(-3+2\omega+3\omega^2\right)\omega^2\right)^{4n+3} +$

$\left(\left(-3+2\omega+3\omega^2\right)^{4n+3}\right) = 0$

$$\Rightarrow \left(-3+2\omega+3\omega^2\right)^{4n+3}\left[\omega^{4n+3}+\omega^{8n+6}+1\right] = 0$$

$$\Rightarrow \ \omega^n + \omega^{2n} + 1 = 0$$

$\Rightarrow$ n is not a multiple of 3.

(D) $\frac{2ab}{a+b} = 4$, $2(5 - a) = b - 5$

$b = 15 - 2a$

$2a(15 - 2a) = 4(15 - a) \Rightarrow 15a - 2a^2$

$= 30 - 2a$

$2a^2 - 17a + 30 = 0$

$\Rightarrow 2a^2 - 12a - 5a + 30 = 0$

$2a(a - 6) - 5(a - 6) = 0$

$$a = \frac{5}{2}, 6$$

$$\Rightarrow |q - a| = |10 - 2a| = 5 \text{ or } 2$$

19. Ans. (4)

Sol. α_k are vertices of 14 sided regular polygon

$|\alpha_{k+1} - \alpha_k|$ length of a side of the regular polygon

$|\alpha_{4k-1} - \alpha_{4k-2}|$ length of a side of the regular polygon

$$\Rightarrow \frac{12(S)}{3(S)} = 4$$

20. Ans. (D)

Sol. Let $p(x) = ax^2 + b + c$

$$p(x) = 0 \Rightarrow x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

so b = 0 as roots are purely imaginary

so equation will be $ax^2 + c = 0$

Now $p(p(x)) = 0$

$$\Rightarrow \quad ap^2(x) + c = 0 \Rightarrow p(x) = \pm\sqrt{-\frac{c}{a}}$$

$$ax^2 + c = \pm\sqrt{-\frac{c}{a}} \ \Rightarrow \ x \notin R$$

if $x = i\beta$ then

$$-a\beta^2 + c = \pm\sqrt{-\frac{c}{a}} \ \text{ not possible}$$

(real) (imaginary)

So, neither real nor purely imaginary roots.

21. Ans. (C)

Sol. (P) $e^{\frac{i2k\pi}{10}} . e^{\frac{i2j\pi}{10}} = 1$

$$e^{i\frac{\pi}{10}\times 2(k+j)} = 1$$

$$\frac{\pi}{10}\left(2\left(k+j\right)\right) = 2n\pi$$

$(k + j) = 10$

Possible

(Q) $e^{\frac{i2\pi}{10}}.z = e^{\frac{i2\pi k}{10}}$

$z = \frac{e^{\frac{i2\pi k}{10}}}{e^{\frac{i2\pi}{10}}}$ is possible

(R) $z^{10} - 1 = (z - 1)(z - z_1)(z - z_2)....(z - z_9)$

put z = 1

$$\lim_{z\to 1}\frac{z^{10}-1}{(z-1)} = (1-z_1)(1-z_2)...(1-z_9)$$

$$\lim_{z\to 1}\frac{10z^9}{1} = (1-z_1)(1-z_2)...(1-z_9)$$

$= |(1 - z_1)(1 - z_2).....(1 - z_9)| = 10$

(S) $1+\cos\frac{2\pi}{10}+\cos\frac{4\pi}{10}+........+\cos\frac{18\pi}{10}=0$

since they are sum of ten, tenth roots of unity

$$\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right) = -1$$

1 + 1 = 2