BINOMIAL THEOREM

1. Let a and b be two nonzero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2 + \frac{70}{27bx}\right)^4$ is equal to the coefficient of x^{-5} is equal to the coefficient of $\left(ax - \frac{1}{bx^2}\right)^7$, then the value of 2b is

[JEE(Advanced) 2023]

2. For non-negative integers s and r, let $\binom{s}{r} = \begin{cases} \frac{s!}{r!(s-r)!} & \text{if } r \le s, \\ 0 & \text{if } r > s. \end{cases}$

For positive integers m and n, let g(m,n) = $\sum_{p=0}^{m+n} \frac{f(m,n,p)}{\binom{n+p}{p}}$

where for any nonnegative integer p, $f(m,n,p) = \sum_{i=0}^{p} \binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}$

Then which of the following statements is/are TRUE? **[JEE(Advanced) 2020]**

(A) g(m, n) = g(n, m) for all positive integers m, n

(B) g(m, n +1) = g(m+1, n) for all positive integers m, n

(C) g(2m, 2n) = 2g(m, n) for all positive integers m, n

(D) g(2m, 2n) = $(g(m, n))^2$ for all positive integers m, n

3. Suppose $\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

[JEE(Advanced) 2019]

4. Let $X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + ... + 10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, r ∈ {1, 2, ..., 10} denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ . **[JEE(Advanced) 2018]**

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE(Advanced) 2016]**

6. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)\ldots..(1 + x^{100})$ is **[JEE(Advanced) 2015]**

7. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is - **[JEE(Advanced) 2014]**

(A) 1051 (B) 1106 (C) 1113 (D) 1120

SOLUTIONS

1. Ans. (3)

Sol. $T_{r+1} = {}^4C_r\left(a.x^2\right)^{4-r}.\left(\frac{70}{27bx}\right)^r$

$= {}^4C_r.a^{4-r}.\frac{70^r}{(27b)^r}.x^{8-3r}$

here $8 - 3r = 5$

$8 - 5 = 3r \Rightarrow r = 1$

$\therefore$ coeff. $= 4.a^3.\frac{70}{27b}$

$T_{r+1} = {}^7C_r(ax)^{7-r}\left(\frac{-1}{bx^2}\right)^r$

$= {}^7C_r.a^{7-r}\left(\frac{-1}{b}\right)^r.x^{7-3r}$

$7 - 3r = -5 \Rightarrow 12 = 3r \Rightarrow r = 4$

coeff. : ${}^7C_4.a^3.\left(\frac{-1}{b}\right)^4 = \frac{35a^3}{b^4}$

now $\frac{35a^3}{b^4} = \frac{280a^3}{27b}$

$b^3 = \frac{35\times 27}{280} = b = \frac{3}{2} \Rightarrow 2b = 3$

2. Ans. (A, B, D)

Sol. Solving

$f(m,n,p) = \sum_{i=0}^{p} {}^mC_i\ {}^{n+i}C_p.{}^{p+n}C_{p-i}$

${}^mC_i.{}^{n+i}C_p.{}^{p+n}C_{p-i}$

${}^mC_i.\frac{(n+i)!}{p!(n-p+i)!}\times\frac{(n+p)!}{(p-i)!(n+i)!}$

${}^mC_i\times\frac{(n+p)!}{p!}\times\frac{1}{(n-p+i)!(p-i)!}$

${}^mC_i\times\frac{(n+p)!}{p!n!}\times\frac{n!}{(n-p+i)!(p-i)!}$

${}^mC_i.{}^{n+p}C_p.{}^nC_{p-i}\quad\left\{{}^mC_i.{}^nC_{p-i} = {}^{m+n}C_p\right\}$

$f(m,n,p) = {}^{n+p}C_p.{}^{m+n}C_p$

$\frac{f(m,n,p)}{{}^{n+p}C_p} = {}^{m+n}C_p$

Now, $g(m,n) = \sum_{p=0}^{m+n}\frac{f(m,n,p)}{{}^{n+p}C_p}$

$g(m,n) = \sum_{p=0}^{m+n}{}^{m+n}C_p$

$g(m, n) = 2^{m+n}$

(A) $g(m, n) = g(n,m)$

(B) $g(m, n + 1) = 2^{m+n+1}$

$g(m + n, n) = 2^{m+1+n}$

(D) $g(2m,2n) = 2^{2m+2n}$

$= (2^{m+n})^2$

$= (g(m,n))^2$

3. Ans. (6.20)

Sol. Suppose

$$\begin{vmatrix}\frac{n(n+1)}{2} & n(n-1).2^{n-2}+n.2^{n-1}\\ n.2^{n-1} & 4^n\end{vmatrix} = 0$$

$\frac{n(n+1)}{2}.4^n - n^2(n-1).2^{2n-3} - n^2 2^{2n-2} = 0$

$\frac{n(n+1)}{2} - \frac{n^2(n-1)}{8} - \frac{n^2}{4} = 0$

$n^2 - 3n - 4 = 0$

$n = 4$

Now $\sum_{k=0}^{4}\frac{{}^4C_k}{k+1} = \sum_{k=0.}^{4}\frac{k+1}{5}.{}^5C_{k+1}\frac{1}{k+1}$

$= \frac{1}{5}.\left[{}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5\right]$

$= \frac{1}{5}\left[2^5 - 1\right] = \frac{31}{5} = 6.20$

4. Ans. (646)

Sol. $X = \sum_{r=0}^{n} r.\left({}^nC_r\right)^2; n = 10$

$X = n.\sum_{r=0}^{n}{}^nC_r.{}^{n-1}C_{r-1}$

$X = n.\sum_{r=1}^{n}{}^nC_{n-r}.{}^{n-1}C_{r-1}$

$X = n.{}^{2n-1}C_{n-1};\ n = 10$

$X = 10.{}^{19}C_9$

$\frac{X}{1430} = \frac{1}{143}.{}^{19}C_9$

$= 646$

5. Ans. (5)

Sol. Coefficient of x^2 in the expansion of

$(1 + x)^2 + (1 + x)^3 + ...(1 + x)^{49} + (1 + mx)^{50}$ is

${}^{2}C_2 + {}^{3}C_2 + ... {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1)\ {}^{51}C_3$

${}^{3}C_3 + {}^{3}C_2 + ... {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1){}^{51}C_3$

${}^{50}C_3 + {}^{50}C_2m^2 = (3n + 1)\ {}^{51}C_3$

$$\frac{50.49.48}{6} + \frac{50.49}{2}m^2 = (3n+1)\ \frac{51.50.49}{6}$$

$m^2 = 51n + 1$

must be a perfect quuared

$\Rightarrow n = 5$ and $m = 16$

6. Ans. (8)

Sol. There are 8 product

$1^{99}\ x^9,\ 1^{98}x\ x^8,\ 1^{98}\ x^2\ x^7,\ 1^{98}\ x^3\ x^6,\ 1^{98}\ x^4\ x^5$

$1^{97}\ x\ x^2\ x^6,\ 1^{97}x\ x^3\ x^5,\ 1^{97}\ x^2\ x^3\ x^4$

which generate x^9 so coeff. is 8

7. Ans. (C)

Sol. Coefficient of x^{11} in

$(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$

$= {}^{4}C_0.{}^{7}C_1.{}^{12}C_2 + {}^{4}C_1.{}^{7}C_3.{}^{12}C_0 + {}^{4}C_2.{}^{7}C_1.{}^{12}C_1 +$
${}^{4}C_4.{}^{7}C_1.{}^{12}C_0$

$= 462 + 140 + 504 + 7 = 1113$