AOD (MONOTONICITY)

1. Let $S = (0, 1) \cup (1, 2) \cup (3, 4)$ and $T = \{0, 1, 2, 3\}$. Then which of the following statements is (are) true ? **[JEE(Advanced) 2023]**

(A) There are infinitely many functions from S to T

(B) There are infinitely many strictly increasing functions from S to T

(C) The number of continuous functions from S to T is at most 120

(D) Every continuous function from S to T is differentiable

2. Let S be the set of all twice differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $\frac{d^2f}{dx^2}(x) > 0$ for all $x \in (-1, 1)$. For $f \in S$, let X_f be the number of points $x\in(-1,1)$ for which $f(x) = x$. Then which of the following statements is(are) true? **[JEE(Advanced) 2023]**

(A) There exists a function $f \in S$ such that $X_f = 0$

(B) For every function $f \in S$, we have $X_f \leq 2$

(C) There exists a function $f \in S$ such that $X_f = 2$

(D) There does **NOT** exist any function f in S such that $X_f = 1$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}.$$

Then which of the following statements is (are) **TRUE** ? **[JEE(Advanced) 2021]**

(A) f is decreasing in the interval (–2,–1)

(B) f is increasing in the interval (1,2)

(C) f is onto

(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

4. For a polynomial g(x) with real coefficient, let m_g denote the number of distinct real roots of g(x). Suppose S is the set of polynomials with real coefficients defined by

$S = \{(x^2 - 1)^2 (a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in R\}$.

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _______ **[JEE(Advanced) 2020]**

5. Let $f : \mathbb{R} \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)? **[JEE(Advanced) 2017]**

(A) $e^x - \int_0^x f(t)\sin t\,dt$

(B) $x^9 - f(x)$

(C) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

(D) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

Answer Q.6, Q.7 and Q.8 by appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x \log_e x$, $x \in (0,\infty)$.

* Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.
* Column 2 contains information about the limiting behavior of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.
* Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0,1)
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0,1)
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

6. Which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**

(A) (IV) (i) (S) (B) (I) (ii) (R) (C) (III) (iv) (P) (D) (II) (iii) (S)

7. Which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**

(A) (III) (iii) (R) (B) (I) (i) (P) (C) (IV) (iv) (S) (D) (II) (ii) (Q)

8. Which of the following options is the only **INCORRECT** combination ? **[JEE(Advanced) 2017]**

(A) (II) (iii) (P) (B) (II) (iv) (Q) (C) (I) (iii) (P) (D) (III) (i) (R)

9. If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then **[JEE(Advanced) 2017]**

(A) $0 < f'(1) \le \frac{1}{2}$

(B) $f'(1) \le 0$

(C) $f'(1) > 1$

(D) $\frac{1}{2} < f'(1) \le 1$

10. Let f, g : [–1, 2] $\to \mathbb{R}$ be continuous function which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table :

	x = –1	x = 0	x = 2
$f(x)$	3	6	0
g(x)	0	1	–1

In each of the intervals (–1, 0) and (0, 2) the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are) **[JEE(Advanced) 2015]**

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in (–1, 0)

(C) $f'(x) - 3g'(x) = 0$ has exactly one solutions in (0, 2)

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

11. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 - 5x + a$. Then **[JEE(Advanced) 2014]**

(A) $f(x)$ has three real roots if $a > 4$

(B) $f(x)$ has only one real roots if $a > 4$

(C) $f(x)$ has three real roots if $a < -4$

(D) $f(x)$ has three real roots if $-4 < a < 4$

SOLUTIONS

1. Ans. (A, C, D)

Sol. $S = (0, 1) \cup (1, 2) \cup (3, 4)$

$T = \{0, 1, 2, 3\}$

Number of functions :

Each element of S have 4 choice

Let n be the number of element in set S.

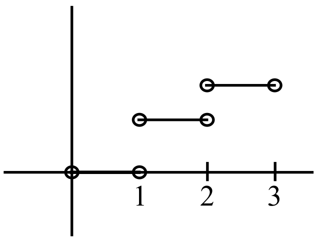
Number of function $= 4^n$

Here $n \to \infty$

$\Rightarrow$ Option (A) is correct.

Option (B) is incorrect (obvious)

(C)



For continuous function

Each interval will have 4 choices.

$\Rightarrow$ Number of continuous functions

$= 4 \times 4 \times 4 = 64$

$\Rightarrow$ Option (C) is correct.

(D) Every continuous function is piecewise constant functions

$\Rightarrow$ Differentiable.

Option (D) is correct.

2. Ans. (A, B, C)

Sol. S = Set of all twice differentiable functions

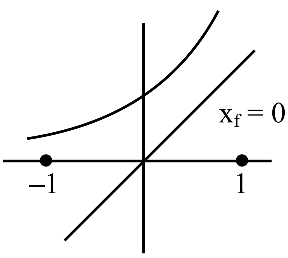
$f : R \to R$

$\frac{d^2f}{dx^2} > 0$ in $(-1, 1)$

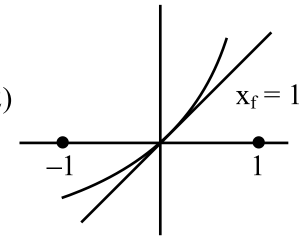
Graph 'f' is Concave upward.

Number of solutions of $f(x) = x \to x_f$

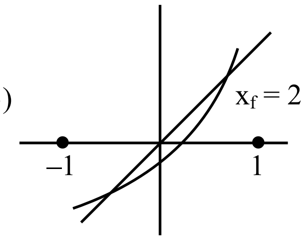
(1)



(2)



(3)



$\Rightarrow$ Graph of $y = f(x)$ can intersect graph of $y = x$ at atmost two points $\Rightarrow 0 \le x_f \le 2$

3. Ans. (A, B)

Sol. $f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$$f'(x) = \frac{(x^2 + 2x + 4)(2x - 3) - (x^2 - 3x - 6)(2x + 2)}{(x^2 + 2x + 4)^2}$$

$$f'(x) = \frac{5x(x + 4)}{(x^2 + 2x + 4)^2}$$

f '(x) : + (–4) – (0) +

$f(-4) = \frac{11}{6}, \quad f(0) = -\frac{3}{2}, \quad \lim_{x \to \pm\infty} f(x) = 1$

Range : $\left[-\frac{3}{2}, \frac{11}{6}\right]$, clearly f(x) is into

4. Ans. (5.00)

Sol. $f(x) = (x^2 - 1)^2 h(x)$;

$h(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

Now, $f(1) = f(-1) = 0$

$\Rightarrow f'(\alpha) = 0, \alpha \in (-1, 1)$ [Rolle's Theorem]

Also, $f'(1) = f'(-1) = 0 \Rightarrow f'(x) = 0$ has at least 3 root, $-1, \alpha, 1$ with $-1 < \alpha < 1$

$\Rightarrow f''(x) = 0$ will have at least 2 root, say β, γ such that

$-1 < \beta < \alpha < \gamma < 1$ [Rolle's Theorem]

So, $\min(m_{f''}) = 2$ and

we find $(m_{f'} + m_{f''}) = 5$ for $f(x) = (x^2 - 1)^2 h(x)$

5. Ans. (B, D)

Sol. For option (A),

Let $g(x) = e^x - \int_0^x f(t)\sin t\,dt$

$\therefore\ g'(x) = e^x - (f(x).\sin x) > 0\ \forall\ x \in (0,1)$

$\Rightarrow$ g(x) is strictly increasing function.

Also, g(0) = 1

$\Rightarrow g(x) > 1\ \forall\ x \in (0,1)$

$\therefore$ option (A) is not possible.

For option (B), let

$k(x) = x^9 - f(x)$

Now, $k(0) = -f(0) < 0$ (As $f \in (0,1)$)

Also, $k(1) = 1 - f(1) > 0$ (As $f \in (0,1)$)

$\Rightarrow k(0).\ k(1) < 0$

So, option(B) is correct.

For option (C), let

$$T(x) = f(x) + \int_0^{\frac{\pi}{2}} f(t).\sin t\,dt$$

$\Rightarrow T(x) > 0\ \forall x \in (0,1)$ (As $f \in (0,1)$)

so, option(C) is not possible.

For option (D),

Let $M(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

$\therefore\ M(0) = 0 - \int_0^{\pi/2} f(t).\cos t\,dt < 0$

Also, $M(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t).\cos t\,dt > 0$

$\Rightarrow M(0).\ M(1) < 0$

$\therefore$ option (D) is correct.

6. Ans. (D)

7. Ans. (D)

8. Ans. (D)

Sol. 6 to 8

$f(x) = x + \ell nx - x\ell nx,\ x > 0$

$f'(x) = \cancel{1} + \frac{1}{x} - \ell n x - \cancel{1}$

$f''(x) = -\frac{1}{x^2} - \frac{1}{x} = \frac{-(x+1)}{x^2}$

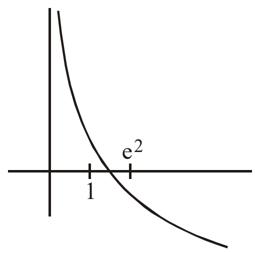
(I) $f(1)\ f(e^2) < 0$ so true

(II) $f'(1)\ f'(e) < 0$ so true

(III) Graph of $f'(x)$ so (III) is false

(IV) Is false

As $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\left[1 + \frac{\ell n x}{x} - \ell n x\right] = -\infty$



$\therefore$ (i) is false (ii) is true

$\lim_{x\to\infty} f'(x) = -\infty$ so (iii) is true

$\lim_{x\to\infty} f''(x) = 0$ so (iv) is true.

(P) $f'(x)$ is positive in (0,1) so true

(Q) $f'(x) < 0$ for in (e, e^2) so true

As $f'(x) < 0\ \forall\ x > 0$ therefor R is false, S is true.

Alternate :

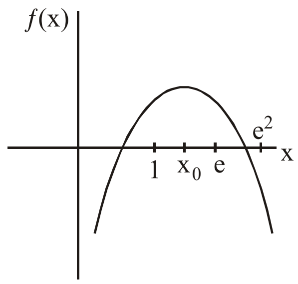
$f(x) = x + \ell nx - x\ell nx$

$f'(x) = \frac{1}{x} - \ell nx = 0$ = at $x = x_0$ where

$x_0 \in (1,e)$

$f''(x) = -\frac{1}{x^2} - \frac{1}{x} < 0\ \forall\ x > 0$

$\Rightarrow f(x)$ concave down



9. Ans. (C)

Sol. Using LMVT on $f(x)$ for $x \in \left[\frac{1}{2}, 1\right]$

$$\frac{f(1)-f\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = f'(c), \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$$\frac{1-\frac{1}{2}}{\frac{1}{2}} = f'(c) \Rightarrow f'(c) = 1, \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$\because$ $f'(x)$ is an increasing function $\forall\ x \in R$

$\therefore$ $f'(1) > 1$

10. Ans. (B, C)

Sol. Let $F(x) = f(x) - 3g(x)$

$\therefore$ $F(-1) = 3$; $F(0) = 3$ & $F(2) = 3$

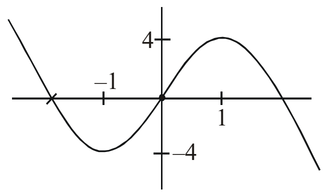
$\therefore$ $F'(x)$ will vanish at least twice in $(-1,0) \cup (0,2)$

$\because$ $F''(x) > 0$ or $< 0\ \forall\ x \in (-1,0) \cup (0,2)$

So there will be exactly 9 solution in $(-1,0)$ and one in $(0,2)$

11. Ans. (B, D)

Sol.



$f(x) = x^5 - 5x + a$

$\therefore$ $a = 5x - x^5$

$\therefore$ $f(x)$ has only one real root if $a > 4$ or $a < -4$

$f(x)$ has three real roots if $-4 < a < 4$