

FINAL JEE(Advanced) EXAMINATION – 2023

(Held On Sunday 04th June, 2023)

PAPER-2

SOLUTION

MATHEMATICS

SECTION-1

1. Ans. (C)

Sol. Diff. wr.t 'x'

$$3f(x) = f(x) + xf(x) - x^2$$

$$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$$

$$IF = e^{-2\int nx dx} = \frac{1}{x^2}$$

$$y\left(\frac{1}{x^2}\right) = \int x \cdot \frac{1}{x^2} dx$$

$$y = x^2 \ln x + cx^2$$

$$\therefore y(1) = \frac{1}{3} \Rightarrow c = \frac{1}{3}$$

$$y(e) = \frac{4e^2}{3}$$

2. Ans. (B)

$$\text{Sol. } P(H) = \frac{1}{3}; P(T) = \frac{2}{3}$$

Req. prob = $P(HH \text{ or } HTHH \text{ or } HTHTHH \text{ or } \dots)$

+ $P(THH \text{ or } THTHH \text{ or } THTHTHH \text{ or } \dots)$

$$= \frac{\frac{1}{3} \cdot \frac{1}{3}}{1 - \frac{2}{3} \cdot \frac{1}{3}} + \frac{\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{1 - \frac{2}{3} \cdot \frac{1}{3}} = \frac{5}{21}$$

3. Ans. (C)

Sol. Case-I : $y \in (-3, 0)$

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \pi + \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$$

$$2\tan^{-1}\left(\frac{6y}{9-y^2}\right) = -\frac{\pi}{3}$$

$$y^2 - 6\sqrt{3}y - 9 = 0 \Rightarrow y = 3\sqrt{3} - 6 \quad (\because y \in (-3, 0))$$

Case-I : $y \in (0, 3)$

$$2 \tan^{-1} \left(\frac{6y}{9-y^2} \right) = \frac{2\pi}{3} \Rightarrow \sqrt{3}y^2 + 6y - 9\sqrt{3} = 0$$

$$y = \sqrt{3} \text{ or } y = -3\sqrt{3} \text{ (rejected)}$$

$$\text{sum} = \sqrt{3} + 3\sqrt{3} - 6 = 4\sqrt{3} - 6$$

4. Ans. (B)

$$\text{Sol. } P(\hat{i} + 2\hat{j} - 5\hat{k}) = P(\vec{a})$$

$$Q(3\hat{i} + 6\hat{j} + 3\hat{k}) = Q(\vec{b})$$

$$R\left(\frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + 7\hat{k}\right) = R(\vec{c})$$

$$S(2\hat{i} + \hat{j} + \hat{k}) = S(\vec{d})$$

$$\frac{\vec{b} + 2\vec{d}}{3} = \frac{7\hat{i} + 8\hat{j} + 5\hat{k}}{3}$$

$$\frac{5\vec{c} + 4\vec{a}}{9} = \frac{21\hat{i} + 24\hat{j} + 15\hat{k}}{9}$$

$$\Rightarrow \frac{\vec{b} + 2\vec{d}}{3} = \frac{5\vec{c} + 4\vec{a}}{9}$$

so [B] is correct.

option -D

$$|\vec{b} \times \vec{d}|^2 = |\vec{b}| |\vec{d}|^2 - (\vec{b} \cdot \vec{d})^2$$

$$= (9 + 36 + 9)(4 + 1 + 1) - (6 + 6 + 3)^2$$

$$= 54 \times 6 - (15)^2$$

$$= 324 - 225$$

$$= 99$$

SECTION-2

5. Ans. (B,C)

$$\text{Sol. } M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$|M| = -1 + 1 = 0 \Rightarrow M$ is singular so non-invertible

$$(B) M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix}$$

$$a_1 + a_2 + a_3 = -a_1$$

$$a_1 + a_3 = -a_2$$

$$a_2 = -a_3$$

$\left. \begin{array}{l} a_1 + a_2 + a_3 = -a_1 \\ a_1 + a_3 = -a_2 \\ a_2 = -a_3 \end{array} \right\} \Rightarrow a_1 = 0 \text{ and } a_2 + a_3 = 0 \text{ infinite solutions exists [B] is correct.}$

Option (D)

$$M - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$|M - 2I| = 0 \Rightarrow [D]$ is wrong

Option (C) :

$$MX = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 0$$

$$x + z = 0$$

$$y = 0$$

\therefore Infinite solution

[C] is correct

6. Ans. (A,B)

$$\text{Sol. } f(x) = \begin{cases} 0 & ; \quad 0 < x < \frac{1}{4} \\ \left(x - \frac{1}{4}\right)^2 \left(x - \frac{1}{2}\right) & ; \quad \frac{1}{4} \leq x < \frac{1}{2} \\ 2\left(x - \frac{1}{4}\right)^2 \left(x - \frac{1}{2}\right) & ; \quad \frac{1}{2} \leq x < \frac{3}{4} \\ 3\left(x - \frac{1}{4}\right)^2 \left(x - \frac{1}{2}\right) & ; \quad \frac{3}{4} \leq x < 1 \end{cases}$$

$f(x)$ is discontinuous at $x = \frac{3}{4}$ only

$$f'(x) = \begin{cases} 0 & ; \quad 0 < x < \frac{1}{4} \\ 2\left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right) + \left(x - \frac{1}{4}\right)^2 & ; \quad \frac{1}{4} < x < \frac{1}{2} \\ 4\left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right) + 2\left(x - \frac{1}{4}\right)^2 & ; \quad \frac{1}{2} < x < \frac{3}{4} \\ 6\left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right) + 3\left(x - \frac{1}{4}\right)^2 & ; \quad \frac{3}{4} < x < 1 \end{cases}$$

$f(x)$ is non-differentiable at $x = \frac{1}{2}$ and $\frac{3}{4}$

minimum values of $f(x)$ occur at $x = \frac{5}{12}$ whose value is $-\frac{1}{432}$

(R)

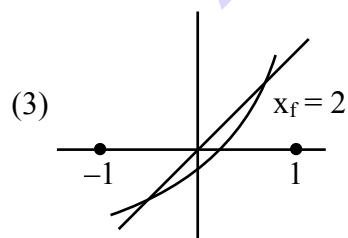
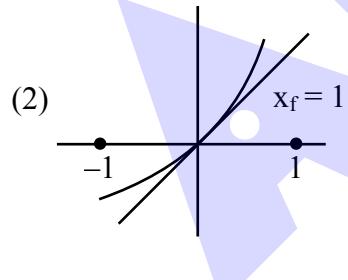
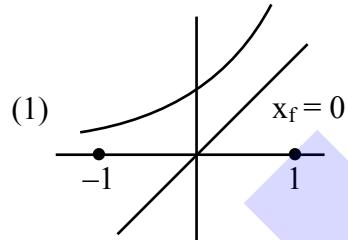
7. Ans. (A,B,C)

Sol. S = Set of all twice differentiable functions $f : R \rightarrow R$

$$\frac{d^2f}{dx^2} > 0 \text{ in } (-1, 1)$$

Graph 'f' is Concave upward.

Number of solutions of $f(x) = x \rightarrow x_f$



\Rightarrow Graph of $y = f(x)$ can intersect graph of $y = x$ at atmost two points $\Rightarrow 0 \leq x_f \leq 2$

Aliter

$$\frac{d^2f(x)}{dx^2} > 0$$

Let $\phi(x) = f(x) - x$

$$\phi''(x) > 0$$

$\therefore \phi'(x) = 0$ has atmost 1 root in $x \in (-1, 1)$

$\therefore \phi(x) = 0$ has atmost 2 roots in $x \in (-1, 1)$

$$\therefore x_f \leq 2$$

SECTION-3

R

8. Ans. (0)

Sol. $f(x) = \int_0^{x \tan^{-1} x} \frac{e^{t-\cos t}}{1+t^{2023}} dt$

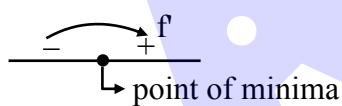
$$f'(x) = \frac{e^{x \tan^{-1} x - \cos(x \tan^{-1} x)}}{1+(x \tan^{-1} x)^{2023}} \cdot \left(\frac{x}{1+x^2} + \tan^{-1} x \right)$$

$$\text{For } x < 0, \tan^{-1} x \in \left(-\frac{\pi}{2}, 0 \right)$$

$$\text{For } x \geq 0, \tan^{-1} x \in \left[0, \frac{\pi}{2} \right]$$

$$\Rightarrow x \tan^{-1} x \geq 0 \quad \forall x \in \mathbb{R}$$

$$\text{And } \frac{x}{1+x^2} + \tan^{-1} x = \begin{cases} > 0 & \text{For } x > 0 \\ < 0 & \text{For } x < 0 \\ 0 & \text{For } x = 0 \end{cases}$$



Hence minimum value is $f(0) = \int_0^0 = 0$

9. Ans. (16)

Sol. $\frac{dy}{dx} - \frac{2x}{x^2 - 5} y = -2x(x^2 - 5)$

$$IF = e^{-\int \frac{2x}{x^2 - 5} dx} = \frac{1}{(x^2 - 5)}$$

$$y \cdot \frac{1}{x^2 - 5} = \int -2x \cdot dx + c$$

$$\Rightarrow \frac{y}{x^2 - 5} = -x^2 + c$$

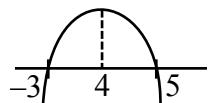
$$x = 2, y = 7$$

$$\frac{7}{-1} = -4 + c \Rightarrow c = -3$$

$$y = -(x^2 - 5)(x^2 + 3)$$

$$\text{put } x^2 = t > 0$$

$$y = -(t - 5)(t + 3)$$



$$y_{\max} = 16 \text{ when } x^2 = 1$$

$$y_{\max} = 16$$

10. Ans. (31)

Sol. No. of elements in X which are multiple of 5

| | |
|--|------------|
| $\overbrace{\quad\quad\quad}^{1,2,2,2} \quad 0 \rightarrow \frac{ 4 }{ 3 } = 4$ $\overbrace{\quad\quad\quad}^{1,4,2,2} \quad 0 \rightarrow \frac{ 4 }{ 2 } = 12$ $\overbrace{\quad\quad\quad}^{4,2,2,2} \quad 0 \rightarrow \frac{ 4 }{ 3 } = 4$ $\overbrace{\quad\quad\quad}^{2,2,4,4} \quad 0 \rightarrow \frac{ 4 }{ 2 2} = 6$ $\overbrace{\quad\quad\quad}^{1,2,4,4} \quad 0 \rightarrow \frac{ 4 }{ 2 } = 12$ | Total = 38 |
|--|------------|

Among these 38 elements, let us calculate when element is not divisible by 20

$$\left. \begin{array}{l} \begin{array}{c} 1 \ 0 \\ \hline 2,2,2 \end{array} \rightarrow \frac{|3|}{|3|} = 1 \\ \begin{array}{c} 1 \ 0 \\ \hline 2,2,4 \end{array} \rightarrow \frac{|3|}{|2|} = 3 \\ \begin{array}{c} 1 \ 0 \\ \hline 2,4,4 \end{array} \rightarrow \frac{|3|}{|2|} = 3 \end{array} \right\} \text{Total} = 7$$

$$\therefore p = \frac{38 - 7}{38} \quad \therefore 38p = 31$$

11. Ans. (512)

Sol. $z^8 - 2^8 = (z - 2)(z - \alpha)(z - \alpha^2) \dots (z - \alpha^7)$

Put $z = 2e^{i\theta}$

$$2^8(e^{i8\theta} - 1) = (2e^{i\theta} - 2)(2e^{i\theta} - \alpha) \dots (2e^{i\theta} - \alpha^7)$$

Take mod

$$2^8|e^{i8\theta} - 1| = PA_1 PA_2 \dots PA_8$$

$$2^8|2\sin 4\theta| = PA_1 PA_2 \dots PA_8$$

$$(PA_1 \cdot PA_2 \dots PA_8)_{\max} = 512$$

12. Ans. (3780)

Sol. Let us calculate when $|R| = 0$

Case-I $ad = bc = 0$

Now $ad = 0$

\Rightarrow Total - (When none of a & d is 0)

$$= 8^2 - 1 = 15 \text{ ways}$$

Similarly $bc = 0 \Rightarrow 15 \text{ ways}$

$$\therefore 15 \times 15 = 225 \text{ ways of } ad = bc = 0$$

Case-II $ad = bc \neq 0$

either $a = d = b = c$ OR $a \neq d, b \neq d$ but $ad = bc$

$${}^7C_1 = 7 \text{ ways}$$

$${}^7C_2 \times 2 \times 2 = 84 \text{ ways}$$

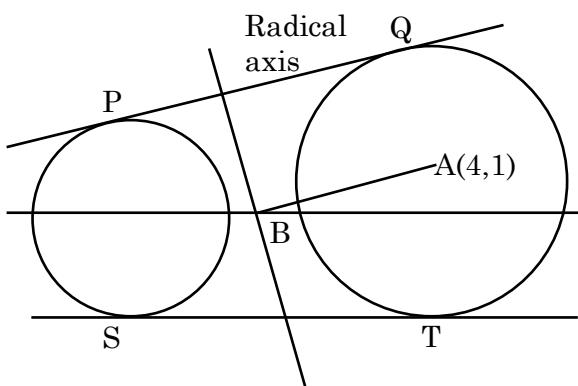
Total 91 ways

$$\therefore |R| = 0 \text{ in } 225 + 91 = 316 \text{ ways}$$

$$|R| \neq 0 \text{ in } 8^4 - 316 = 3780$$

13. Ans. (2)

Sol.



$$\text{Let } C_2: (x - 4)^2 + (y - 1)^2 = r^2$$

$$\text{radical axis } 8x + 2y - 17 = 1 - r^2$$

$$8x + 2y = 18 - r^2$$

$$B\left(\frac{18-r^2}{8}, 0\right) \quad A(4,1)$$

$$AB = \sqrt{5}$$

$$\sqrt{\left(\frac{18-r^2}{8} - 4\right)^2 + 1} = \sqrt{5}$$

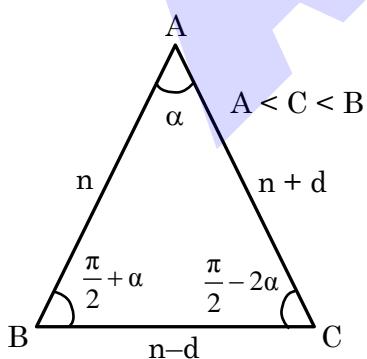
$$r^2 = 2$$

$$\Rightarrow n = \sin\alpha + \cos\alpha$$

SECTION-4

14. Ans. (1007.99 to 1008.01)

Sol.



$$n - d = 2 \sin \alpha \quad \dots(1)$$

$$n + d = 2 \sin \left(\frac{\pi}{2} + \alpha \right)$$

$$\Rightarrow n + d = 2 \cos \alpha \quad \dots(2)$$

$$n = 2 \sin \left(\frac{\pi}{2} - 2\alpha \right)$$

$$\Rightarrow n = 2 \cos 2\alpha \quad \dots(3)$$

$$\Rightarrow 2 \cos 2\alpha = \sin \alpha + \cos \alpha$$

$$\Rightarrow 2(\cos \alpha - \sin \alpha) = 1$$

$$\Rightarrow \sin 2\alpha = \frac{3}{4}$$

$$\text{Then, } a = \frac{1}{2} \cdot n \cdot (n + d) \cdot \sin \alpha = \frac{1}{2} \cdot 2 \cos 2\alpha \cdot 2 \cos \alpha \cdot \sin \alpha$$

$$= \sin 2\alpha \cdot \cos 2\alpha$$

$$= \frac{3}{4} \times \frac{\sqrt{7}}{4} = \frac{3\sqrt{7}}{16}$$

$$(64a)^2 = \left(64 \times \frac{3\sqrt{7}}{16} \right)^2 = 16 \times 9 \times 7 = 1008$$

15. Ans. (0.24 to 0.26)

Sol. From above equation in Ques. 14

$$r = \frac{\Delta}{s} = \frac{1}{2} \frac{n(n+d)\sin \alpha}{\left(\frac{3n}{2} \right)}$$

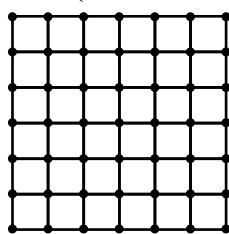
$$= \frac{(n+d)\sin \alpha}{3}$$

$$= \frac{2 \cos \alpha \sin \alpha}{3} \quad (\text{from (2)})$$

$$r = \frac{\sin 2\alpha}{3} = \frac{1}{4}$$

16. Ans. (23.99 to 24.01)

Sol.



P_i = Probability that randomly selected points has friends

$P_0 = 0$ (0 friends)

$P_1 = 0$ (exactly 1 friends)

$$P_2 = \frac{^4C_1}{^{49}C_1} = \frac{4}{9} \text{ (exactly 2 friends)}$$

$$P_3 = \frac{^{20}C_1}{^{49}C_1} = \frac{20}{49} \text{ (exactly 3 friends)}$$

$$P_4 = \frac{^{25}C_1}{^{49}C_1} = \frac{25}{49} \text{ (exactly 4 friends)}$$

| | | | | | |
|--------|---|---|----------------|-----------------|-----------------|
| x | 0 | 1 | 2 | 3 | 4 |
| $P(x)$ | 0 | 0 | $\frac{4}{49}$ | $\frac{20}{49}$ | $\frac{25}{49}$ |

$$\text{Mean} = E(x) = \sum x_i P_i = 0 + 0 + \frac{8}{49} + \frac{60}{49} + \frac{100}{49} = \frac{168}{49}$$

$$7(E(x)) = \frac{168}{49} \times 7 = 24$$

17. Ans. (0.49 to 0.51)

Sol. Total number of ways of selecting 2 persons = $^{49}C_2$

Number of ways in which 2 friends are selected = $6 \times 7 \times 2 = 84$

$$7P = \frac{84 \times 2}{49 \times 48} \times 7 = \frac{1}{2}$$