

FINAL JEE(Advanced) EXAMINATION – 2023**(Held On Sunday 04th June, 2023)****PAPER-1****TEST PAPER****MATHEMATICS****SECTION-1 : (Maximum Marks : 12)**

- This section contains **THREE (03)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 **ONLY** if (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all the four options are correct but **ONLY** three options are chosen;

Partial Marks : +2 If three or more options are correct but **ONLY** two options are chosen, both of which are correct;

Partial Marks : +1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : -2 In all other cases.

- For example, in a question, if (A), (B) and (D) are the **ONLY** three options corresponding to correct answers, then
choosing **ONLY** (A), (B) and (D) will get +4 marks;
choosing **ONLY** (A) and (B) will get +2 marks;
choosing **ONLY** (A) and (D) will get +2 marks;
choosing **ONLY** (B) and (D) will get +2 marks;
choosing **ONLY** (A) will get +1 marks;
choosing **ONLY** (B) will get +1 marks;
choosing **ONLY** (D) will get +1 marks;
choosing no option (i.e. the question is unanswered) will get 0 marks; and
choosing any other combination of options will get -2 marks.

1. Let $S = (0, 1) \cup (1, 2) \cup (3, 4)$ and $T = \{0, 1, 2, 3\}$. Then which of the following statements is(are) true ?
 - (A) There are infinitely many functions from S to T
 - (B) There are infinitely many strictly increasing functions from S to T
 - (C) The number of continuous functions from S to T is at most 120
 - (D) Every continuous function from S to T is differentiable

2. Let T_1 and T_2 be two distinct common tangents to the ellipse $E : \frac{x^2}{6} + \frac{y^2}{3} = 1$ and the parabola $P : y^2 = 12x$. Suppose that the tangent T_1 touches P and E at the point A_1 and A_2 , respectively and the tangent T_2 touches P and E at the points A_4 and A_3 , respectively. Then which of the following statements is(are) true?
 - (A) The area of the quadrilateral $A_1A_2A_3A_4$ is 35 square units
 - (B) The area of the quadrilateral $A_1A_2A_3A_4$ is 36 square units
 - (C) The tangents T_1 and T_2 meet the x -axis at the point $(-3, 0)$
 - (D) The tangents T_1 and T_2 meet the x -axis at the point $(-6, 0)$

3. Let $f : [0, 1] \rightarrow [0, 1]$ be the function defined by $f(x) = \frac{x^3}{3} - x^2 + \frac{5}{9}x + \frac{17}{36}$. Consider the square region $S = [0, 1] \times [0, 1]$. Let $G = \{(x, y) \in S : y > f(x)\}$ be called the green region and $R = \{(x, y) \in S : y < f(x)\}$ be called the red region. Let $L_h = \{(x, h) \in S : x \in [0, 1]\}$ be the horizontal line drawn at a height $h \in [0, 1]$. Then which of the following statements is(are) true?
 - (A) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the green region below the line L_h
 - (B) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the red region below the line L_h
 - (C) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the red region below the line L_h
 - (D) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the green region below the line L_h

SECTION-2 : (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +3 If **ONLY** the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : -1 In all other cases.

4. Let $f : (0, 1) \rightarrow \mathbb{R}$ be the functions defined as $f(x) = \sqrt{n}$ if $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$ where $n \in \mathbb{N}$. Let $g : (0, 1) \rightarrow \mathbb{R}$ be a function such that $\int_{x^2}^x \sqrt{\frac{1-t}{t}} dt < g(x) < 2\sqrt{x}$ for all $x \in (0, 1)$. Then $\lim_{x \rightarrow 0} f(x)g(x)$
- (A) does **NOT** exist (B) is equal to 1
(C) is equal to 2 (D) is equal to 3
5. Let Q be the cube with the set of vertices $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2, x_3 \in \{0, 1\}\}$. Let F be the set of all twelve lines containing the diagonals of the six faces of the cube Q . Let S be the set of all four lines containing the main diagonals of the cube Q ; for instance, the line passing through the vertices $(0, 0, 0)$ and $(1, 1, 1)$ is in S . For lines ℓ_1 and ℓ_2 , let $d(\ell_1, \ell_2)$ denote the shortest distance between them. Then the maximum value of $d(\ell_1, \ell_2)$, as ℓ_1 varies over F and ℓ_2 varies over S , is
- (A) $\frac{1}{\sqrt{6}}$ (B) $\frac{1}{\sqrt{8}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{12}}$
6. Let $X = \left\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} : \frac{x^2}{8} + \frac{y^2}{20} < 1 \text{ and } y^2 < 5x \right\}$. Three distinct points P , Q and R are randomly chosen from X . Then the probability that P , Q and R form a triangle whose area is a positive integer, is
- (A) $\frac{71}{220}$ (B) $\frac{73}{220}$ (C) $\frac{79}{220}$ (D) $\frac{83}{220}$
7. Let P be a point on the parabola $y^2 = 4ax$, where $a > 0$. The normal to the parabola at P meets the x -axis at a point Q . The area of the triangle PFQ , where F is the focus of the parabola, is 120. If the slope m of the normal and a are both positive integers, then the pair (a, m) is
- (A) $(2, 3)$ (B) $(1, 3)$ (C) $(2, 4)$ (D) $(3, 4)$

SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +4 **ONLY** If the correct integer is entered;
Zero Marks : 0 In all other cases.

8. Let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for $x \in \mathbb{R}$. Then the number of real solutions of the equation

$$\sqrt{1 + \cos(2x)} = \sqrt{2} \tan^{-1}(\tan x) \text{ in the set } \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \text{ is equal to}$$

9. Let $n \geq 2$ be a natural number and $f : [0, 1] \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} n(1 - 2nx) & \text{if } 0 \leq x \leq \frac{1}{2n} \\ 2n(2nx - 1) & \text{if } \frac{1}{2n} \leq x \leq \frac{3}{4n} \\ 4n(1 - nx) & \text{if } \frac{3}{4n} \leq x \leq \frac{1}{n} \\ \frac{n}{n-1}(nx - 1) & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

If n is such that the area of the region bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = f(x)$ is 4, then the maximum value of the function f is

10. Let $75\dots 57$ denote the $(r + 2)$ digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + \dots + 75\dots 57$. If $S = \frac{75\dots 57 + m}{n}$, where m and n are natural numbers less than 3000, then the value of $m + n$ is

11. Let $A = \left\{ \frac{1967 + 1686i \sin \theta}{7 - 3i \cos \theta} : \theta \in \mathbb{R} \right\}$. If A contains exactly one positive integer n , then the value of n is

12. Let P be the plane $\sqrt{3}x + 2y + 3z = 16$ and let

$$S = \left\{ \alpha \hat{i} + \beta \hat{j} + \gamma \hat{k} : \alpha^2 + \beta^2 + \gamma^2 = 1 \text{ and the distance of } (\alpha, \beta, \gamma) \text{ from the plane } P \text{ is } \frac{7}{2} \right\}.$$

Let \vec{u} , \vec{v} and \vec{w} be three distinct vectors in S such that $|\vec{u} - \vec{v}| = |\vec{v} - \vec{w}| = |\vec{w} - \vec{u}|$. Let V be the volume of the parallelepiped determined by vectors \vec{u} , \vec{v} and \vec{w} . Then the value of $\frac{80}{\sqrt{3}} V$ is

13. Let a and b be two nonzero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2 + \frac{70}{27bx}\right)^4$ is equal to the coefficient of x^{-5} is equal to the coefficient of $\left(ax - \frac{1}{bx^2}\right)^7$, then the value of $2b$ is

SECTION-4 : (Maximum Marks : 12)

- This section contains **FOUR (04)** Matching List Sets.
- Each set has **ONE** Multiple Choice Question.
- Each set has **TWO** lists : **List-I** and **List-II**.
- **List-I** has **Four** entries (P), (Q), (R) and (S) and **List-II** has **Five** entries (1), (2), (3), (4) and (5).
- **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +3 **ONLY** if the option corresponding to the correct combination is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : -1 In all other cases.

14. Let α , β and γ be real numbers. consider the following system of linear equations

$$x + 2y + z = 7$$

$$x + \alpha z = 11$$

$$2x - 3y + \beta z = \gamma$$

Match each entry in **List - I** to the correct entries in **List-II**

List-I

List-II

(P) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma = 28$, then the system has

(1) a unique solution

(Q) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma \neq 28$, then the system has

(2) no solution

(R) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma \neq 28$,

(3) infinitely many solutions

then the system has

(S) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma = 28$,

(4) $x = 11$, $y = -2$ and $z = 0$ as a solution

then the system has

(5) $x = -15$, $y = 4$ and $z = 0$ as a solution

The correct option is :

- (A) (P) \rightarrow (3) (Q) \rightarrow (2) (R) \rightarrow (1) (S) \rightarrow (4)
 (B) (P) \rightarrow (3) (Q) \rightarrow (2) (R) \rightarrow (5) (S) \rightarrow (4)
 (C) (P) \rightarrow (2) (Q) \rightarrow (1) (R) \rightarrow (4) (S) \rightarrow (5)
 (D) (P) \rightarrow (2) (Q) \rightarrow (1) (R) \rightarrow (1) (S) \rightarrow (3)

15. Consider the given data with frequency distribution

x_i	3	8	11	10	5	4
f_i	5	2	3	2	4	4

Match each entry in **List-I** to the correct entries in **List-II**.

List-I

- (P) The mean of the above data is
(Q) The median of the above data is
(R) The mean deviation about the mean of the above data is
(S) The mean deviation about the median of the above data is

List-II

- (1) 2.5
(2) 5
(3) 6
(4) 2.7
(5) 2.4

The correct option is :

- (A) (P) \rightarrow (3) (Q) \rightarrow (2) (R) \rightarrow (4) (S) \rightarrow (5)
(B) (P) \rightarrow (3) (Q) \rightarrow (2) (R) \rightarrow (1) (S) \rightarrow (5)
(C) (P) \rightarrow (2) (Q) \rightarrow (3) (R) \rightarrow (4) (S) \rightarrow (1)
(D) (P) \rightarrow (3) (Q) \rightarrow (3) (R) \rightarrow (5) (S) \rightarrow (5)

16. Let ℓ_1 and ℓ_2 be the lines $\vec{r}_1 = \lambda(\hat{i} + \hat{j} + \hat{k})$ and $\vec{r}_2 = (\hat{j} - \hat{k}) + \mu(\hat{i} + \hat{k})$, respectively. Let X be the set of all the planes H that contain the line ℓ_1 . For a plane H, let $d(H)$ denote the smallest possible distance between the points of ℓ_2 and H. Let H_0 be plane in X for which $d(H_0)$ is the maximum value of $d(H)$ as H varies over all planes in X.

Match each entry in **List-I** to the correct entries in **List-II**.

List-I

- (P) The value of $d(H_0)$ is
(Q) The distance of the point (0,1,2) from H_0 is
(R) The distance of origin from H_0 is
(S) The distance of origin from the point of intersection of planes $y = z$, $x = 1$ and H_0 is

List-II

- (1) $\sqrt{3}$
(2) $\frac{1}{\sqrt{3}}$
(3) 0
(4) $\sqrt{2}$
(5) $\frac{1}{\sqrt{2}}$

The correct option is :

- (A) (P) \rightarrow (2) (Q) \rightarrow (4) (R) \rightarrow (5) (S) \rightarrow (1)
(B) (P) \rightarrow (5) (Q) \rightarrow (4) (R) \rightarrow (3) (S) \rightarrow (1)
(C) (P) \rightarrow (2) (Q) \rightarrow (1) (R) \rightarrow (3) (S) \rightarrow (2)
(D) (P) \rightarrow (5) (Q) \rightarrow (1) (R) \rightarrow (4) (S) \rightarrow (2)

17. Let z be complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where \bar{z} denotes the complex conjugate of z . Let the imaginary part of z be nonzero.

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	List-II
(P) $ z ^2$ is equal to	(1) 12
(Q) $ z - \bar{z} ^2$ is equal to	(2) 4
(R) $ z ^2 + z + \bar{z} ^2$ is equal to	(3) 8
(S) $ z + 1 ^2$ is equal to	(4) 10
	(5) 7

The correct option is :

- (A) (P) \rightarrow (1) (Q) \rightarrow (3) (R) \rightarrow (5) (S) \rightarrow (4)
 (B) (P) \rightarrow (2) (Q) \rightarrow (1) (R) \rightarrow (3) (S) \rightarrow (5)
 (C) (P) \rightarrow (2) (Q) \rightarrow (4) (R) \rightarrow (5) (S) \rightarrow (1)
 (D) (P) \rightarrow (2) (Q) \rightarrow (3) (R) \rightarrow (5) (S) \rightarrow (4)