

FINAL JEE(Advanced) EXAMINATION - 2022

(Held On Sunday 28th AUGUST, 2022)

PAPER-2

TEST PAPER WITH SOLUTION

MATHEMATICS

SECTION-1 : (Maximum Marks : 24)

- This section contains **EIGHT (08)** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from **0 TO 9, BOTH INCLUSIVE**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

<i>Full Marks</i>	: +3	If ONLY the correct integer is entered;
<i>Zero Marks</i>	: 0	If the question is unanswered;
<i>Negative Marks</i>	: -1	In all other cases.

1. Let α and β be real numbers such that $-\frac{\pi}{4} < \beta < 0 < \alpha < \frac{\pi}{4}$. If $\sin(\alpha + \beta) = \frac{1}{3}$ and $\cos(\alpha - \beta) = \frac{2}{3}$, then the greatest integer less than or equal to

$$\left(\frac{\sin \alpha}{\cos \beta} + \frac{\cos \beta}{\sin \alpha} + \frac{\cos \alpha}{\sin \beta} + \frac{\sin \beta}{\cos \alpha} \right)^2$$

is _____.

Ans. 1

Sol. $\alpha \in \left(0, \frac{\pi}{4}\right), \beta \in \left(-\frac{\pi}{4}, 0\right) \Rightarrow \alpha + \beta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$$\sin(\alpha + \beta) = \frac{1}{3}, \cos(\alpha - \beta) = \frac{2}{3}$$

$$\left(\frac{\sin \alpha}{\cos \beta} + \frac{\cos \alpha}{\sin \beta} + \frac{\cos \beta}{\sin \alpha} + \frac{\sin \beta}{\cos \alpha} \right)^2$$

$$\left(\frac{\cos(\alpha - \beta)}{\cos \beta \sin \beta} + \frac{\cos(\beta - \alpha)}{\sin \alpha \cos \alpha} \right)^2$$

$$= 4 \cos^2(\alpha - \beta) \left(\frac{1}{\sin 2\beta} + \frac{1}{\sin 2\alpha} \right)^2$$

$$= 4 \cos^2(\alpha - \beta) \left(\frac{2 \sin(\alpha + \beta) \cos(\alpha - \beta)}{\sin 2\alpha \sin 2\beta} \right) \quad \dots(1)$$

$$\begin{aligned}
 &= \frac{16 \cos^4(\alpha - \beta) \sin^2(\alpha + \beta) \times 4}{(\cos 2(\alpha - \beta) - \cos 2(\alpha + \beta))^2} \\
 &= \frac{64 \cos^4(\alpha - \beta) \sin^2(\alpha + \beta)}{(2 \cos^2(\alpha - \beta) - 1 - 1 + 2 \sin^2(\alpha + \beta))^2} \\
 &= 64 \times \frac{16}{81} \times \frac{1}{9 \left(2 \times \frac{4}{9} - 1 - 1 + \frac{2}{9} \right)^2} \\
 &= \frac{64 \times 16}{81 \times 9} \cdot \frac{81}{64} = \frac{16}{9} \\
 \left[\frac{16}{9} \right] &= 1 \text{ Ans.}
 \end{aligned}$$

2. If $y(x)$ is the solution of the differential equation

$$x dy - (y^2 - 4y) dx = 0 \text{ for } x > 0, y(1) = 2,$$

and the slope of the curve $y = y(x)$ is never zero, then the value of $10y(\sqrt{2})$ is _____.

Ans. 8

Sol. $x dy - (y^2 - 4y) dx = 0, x > 0$

$$\int \frac{dy}{y^2 - 4y} = \int \frac{dx}{x}$$

$$\int \left(\frac{1}{y-4} - \frac{1}{y} \right) dy = 4 \int \frac{dx}{x}$$

$$\log_e |y-4| - \log_e |y| = 4 \log_e x + \log_e c$$

$$\frac{|y-4|}{|y|} = cx^4 \xrightarrow{(1,2)} c = 1$$

$$|y-4| = |y|x^4$$

C-1 and C-2

$$y - 4 = yx^4$$

$$y - 4 = -yx^4$$

$$y = \frac{4}{1-x^4}$$

$$y = \frac{4}{1+x^4}$$

$$y(1) = \text{ND (rejected)}$$

$$y(1) = 2$$

$$y(\sqrt{2}) = \frac{4}{5} \Rightarrow 10y(\sqrt{2}) = 8$$

3. The greatest integer less than or equal to

$$\int_1^2 \log_2(x^3 + 1) dx + \int_1^{\log_2 9} (2^x - 1)^{\frac{1}{3}} dx$$

is _____.

Ans. 5

Sol. $f(x) = \log_2(x^3 + 1) = y$

$$x^3 + 1 = 2^y \Rightarrow x = (2^y - 1)^{1/3} = f^{-1}(y)$$

$$f^{-1}(x) = (2^x - 1)^{1/3}$$

$$= \int_1^2 \log_2(x^3 + 1) dx + \int_1^{\log_2 9} (2^x - 1)^{1/3} dx$$

$$= \int_1^2 f(x) dx + \int_1^{\log_2 9} f^{-1}(x) dx = 2 \log_2 9 - 1$$

$$= 8 < 9 < 2^{7/2} \Rightarrow 3 < \log_2 9 < \frac{7}{2}$$

$$= 5 < 2 \log_2 9 - 1 < 6$$

$$[2 \log_2 9 - 1] = 5$$

4. The product of all positive real values of x satisfying the equation

$$x^{(16(\log_5 x)^3 - 68 \log_5 x)} = 5^{-16}$$

is _____.

Ans. 1

Sol. $x^{16(\log_5 x)^3 - 68 \log_5 x} = 5^{-16}$

Take log to the base 5 on both sides and put $\log_5 x = t$

$$16t^4 - 68t^2 + 16 = 0$$

$$\Rightarrow 4t^4 - 17t^2 + 4 = 0 \left\{ \begin{array}{l} t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \right.$$

$$t_1 + t_2 + t_3 + t_4 = 0$$

$$\log_5 x_1 + \log_5 x_2 + \log_5 x_3 + \log_5 x_4 = 0$$

$$x_1 x_2 x_3 x_4 = 1$$

5. If

$$\beta = \lim_{x \rightarrow 0} \frac{e^{x^3} - (1-x^3)^{\frac{1}{3}} + \left((1-x^2)^{\frac{1}{2}} - 1 \right) \sin x}{x \sin^2 x}$$

then the value of 6β is _____.

Ans. 5

Sol.
$$\beta = \lim_{x \rightarrow 0} \frac{e^{x^3} - (1-x^3)^{1/3} + \left((1-x^2)^{1/2} - 1 \right) \sin x}{\frac{x \sin^2 x}{x^2} x^2} + \frac{\left((1-x^2)^{1/2} - 1 \right) \sin x}{x \frac{\sin^2 x}{x^2} x^2}$$

use expansion

$$\beta = \lim_{x \rightarrow 0} \frac{(1+x^3) - \left(1 - \frac{x^3}{3}\right)}{x^3} + \lim_{x \rightarrow 0} \frac{\left(\left(1 - \frac{x^2}{2}\right) - 1 \right) \frac{\sin x}{x}}{x^2}$$

$$\beta = \lim_{x \rightarrow 0} \frac{4x^3}{3x^3} + \lim_{x \rightarrow 0} \frac{-x^2}{2x^2}$$

$$\beta = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$$

$$6\beta = 5$$

6. Let β be a real number. Consider the matrix

$$A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$$

If $A^7 - (\beta - 1)A^6 - \beta A^5$ is a singular matrix, then the value of 9β is _____.

Ans. 3

Sol.
$$A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix} \quad |A| = -1$$

$$\Rightarrow |A^7 - (\beta - 1)A^6 - \beta A^5| = 0$$

$$\Rightarrow |A|^5 |A^2 - (\beta - 1)A - \beta I| = 0$$

$$\Rightarrow |A|^5 |(A^2 - \beta A) + A - \beta I| = 0$$

$$\Rightarrow |A|^5 |A(A - \beta I) + I(A - \beta I)| = 0$$

$$|A|^5 |(A + I)(A - \beta I)| = 0$$

$$A + I = \begin{pmatrix} \beta+1 & 0 & 1 \\ 2 & 2 & -2 \\ 3 & 1 & -1 \end{pmatrix} \Rightarrow |A + I| = -4, \text{ Here } |A| \neq 0 \text{ \& } |A + I| \neq 0$$

$$A - \beta I = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1-\beta & -2 \\ 3 & 1 & -2-\beta \end{pmatrix}$$

$$|A - \beta I| = 2 - 3(1 - \beta) = 3\beta - 1 = 0 \Rightarrow \beta = \frac{1}{3}$$

$$9\beta = 3$$

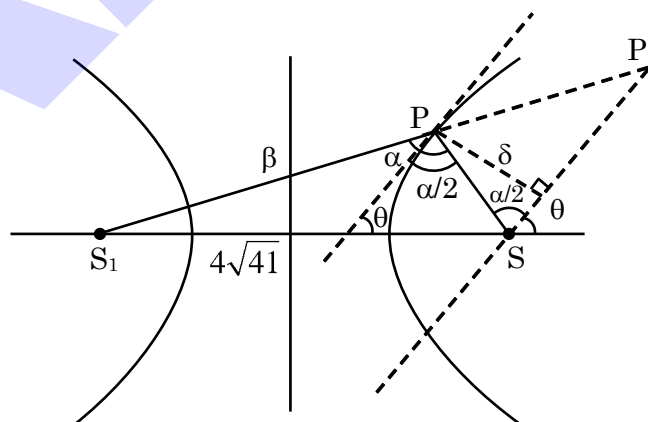
7. Consider the hyperbola

$$\frac{x^2}{100} - \frac{y^2}{64} = 1$$

with foci at S and S_1 , where S lies on the positive x -axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1 = \alpha$, with $\alpha < \frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola, intersects the straight line S_1P at P_1 . Let δ be the distance of P from the straight line SP_1 , and $\beta = S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9} \sin \frac{\alpha}{2}$ is _____.

Ans. 7

Sol.



$$S_1P - SP = 20$$

$$\beta - \frac{\delta}{\sin \frac{\alpha}{2}} = 20$$

$$\beta^2 + \frac{\delta^2}{\sin^2 \frac{\alpha}{2}} - 400 = \frac{2\beta\delta}{\sin \frac{\alpha}{2}}$$

$$\frac{1}{SP} = \frac{\sin \frac{\alpha}{2}}{\delta}$$

$$\cos \alpha = \frac{SP^2 + \beta^2 - 656}{2\beta \frac{\delta}{\sin \frac{\alpha}{2}}}$$

$$\frac{\frac{2\beta\delta}{\sin \frac{\alpha}{2}} - 256}{\frac{2\beta\delta}{\sin \frac{\alpha}{2}}} = \cos \alpha$$

$$\frac{\lambda - 128}{\lambda} = \cos \alpha$$

$$\lambda(1 - \cos \alpha) = 128$$

$$\frac{\beta\delta}{\sin \frac{\alpha}{2}} \cdot 2 \sin^2 \frac{\alpha}{2} = 128$$

$$\frac{\beta\delta}{9} \sin \frac{\alpha}{2} = \frac{64}{9} \Rightarrow \left[\frac{\beta\delta}{9} \sin \frac{\alpha}{2} \right] = 7 \text{ where } [.] \text{ denotes greatest integer function}$$

8. Consider the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = x^2 + \frac{5}{12} \text{ and } g(x) = \begin{cases} 2\left(1 - \frac{4|x|}{3}\right), & |x| \leq \frac{3}{4}, \\ 0, & |x| > \frac{3}{4}. \end{cases}$$

If α is the area of the region

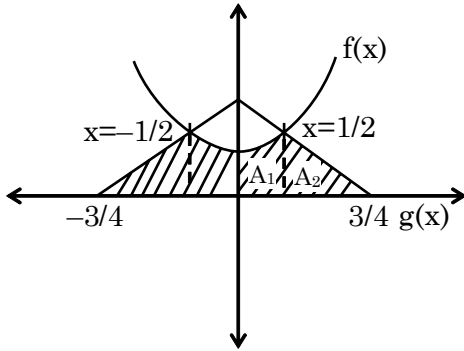
$$\left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq \frac{3}{4}, 0 \leq y \leq \min\{f(x), g(x)\} \right\},$$

then the value of 9α is _____.

Ans. 6

Sol. $x^2 + \frac{5}{12} = \frac{2-8x}{3}$

$$x^2 + \frac{8x}{3} + \frac{5}{12} - 2 = 0$$



$$12x^2 + 32x - 19 = 0$$

$$12x^2 + 38x - 6x - 19 = 0$$

$$2x(6x + 19) - 1(6x + 19) = 0$$

$$(6x + 19)(2x - 1) = 0$$

$$\boxed{x = \frac{1}{2}}$$

$$\alpha = 2A_1 + A_2$$

$$\alpha = 2 \left(\int_0^{1/2} x^2 + \frac{5}{12} dx + \frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} \right)$$

$$\Rightarrow \alpha = 2 \left[\left(\frac{x^3}{3} + \frac{5x}{12} \right)_0^{1/2} + \frac{1}{12} \right]$$

$$\Rightarrow \alpha = 2 \left[\frac{1}{24} + \frac{5}{24} + \frac{1}{12} \right]$$

$$\Rightarrow \alpha = 2 \left[\frac{1+5+2}{24} \right] \Rightarrow \alpha = 2 \times \frac{8}{24} \Rightarrow 9\alpha = 9 \times \frac{8}{12}$$

$$\Rightarrow 9\alpha = 6$$

Apply sine-rule in ΔPRQ :

$$\frac{\alpha}{\sin 30^\circ} = \frac{1}{\sin 80^\circ} \Rightarrow \alpha = \frac{1}{2 \sin 80^\circ} \quad \dots(1)$$

Apply sine-rule in ΔPRS

$$\frac{\beta}{\sin 40^\circ} = \frac{1}{\sin \theta} \Rightarrow \beta \sin \theta = \sin 40^\circ \quad \dots(2)$$

$$4\alpha\beta \sin \theta = \frac{4 \sin 40^\circ}{2 \sin 80^\circ} = \frac{4 \sin 40^\circ}{2(2 \sin 40^\circ \cos 40^\circ)}$$

$$= \sec 40^\circ$$

Now $\sec 30^\circ < \sec 40^\circ < \sec 45^\circ$

$$\Rightarrow \frac{2}{\sqrt{3}} < \sec 40^\circ < \sqrt{2}$$

10. Let

$$\alpha = \sum_{k=1}^{\infty} \sin^{2k} \left(\frac{\pi}{6} \right).$$

Let $g : [0, 1] \rightarrow \mathbb{R}$ be the function defined by

$$g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}$$

Then, which of the following statements is/are TRUE?

- (A) The minimum value of $g(x)$ is $2^{\frac{7}{6}}$
- (B) The maximum value of $g(x)$ is $1 + 2^{\frac{1}{3}}$
- (C) The function $g(x)$ attains its maximum at more than one point
- (D) The function $g(x)$ attains its minimum at more than one point

Ans. (A,B,C)

Sol. $\alpha = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \dots$

$$\alpha = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$$

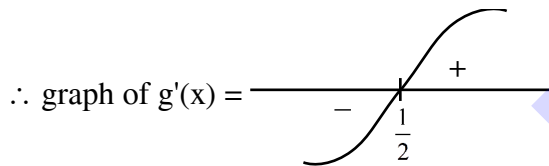
$$\therefore g(x) = 2^{x/3} + 2^{1/3(1-x)}$$

$$\therefore g(x) = 2^{x/3} + \frac{2^{1/3}}{2^{x/3}}$$

where $g(0) = 1 + 2^{1/3}$ & $g(1) = 1 + 2^{1/3}$

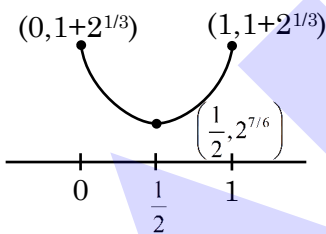
$$\therefore g'(x) = \frac{1}{3} \left(2^{x/3} - \frac{2^{1/3}}{2^{x/3}} \right) = 0$$

$$\Rightarrow 2^{2x/3} = 2^{1/3} \Rightarrow x = \frac{1}{2} = \text{critical point}$$



& $g\left(\frac{1}{2}\right) = 2^{7/6}$

\therefore graph of $g(x)$ in $[0, 1]$



11. Let \bar{z} denote the complex conjugate of a complex number z . If z is a non-zero complex number for which both real and imaginary parts of

$$(\bar{z})^2 + \frac{1}{z^2}$$

are integers, then which of the following is/are possible value(s) of $|z|$?

(A) $\left(\frac{43 + 3\sqrt{205}}{2}\right)^{\frac{1}{4}}$

(B) $\left(\frac{7 + \sqrt{33}}{4}\right)^{\frac{1}{4}}$

(C) $\left(\frac{9 + \sqrt{65}}{4}\right)^{\frac{1}{4}}$

(D) $\left(\frac{7 + \sqrt{13}}{6}\right)^{\frac{1}{4}}$

Ans. (A)

Sol. Let $(\bar{z})^2 + \frac{1}{z^2} = m + in$, $m, n \in \mathbb{Z}$

$$(\bar{z})^2 + \frac{\bar{z}^2}{|z|^4} = m + in$$

$$\Rightarrow (x^2 - y^2) \left(1 + \frac{1}{|z|^4} \right) = m \quad \dots(1)$$

$$\& -2xy \left(1 + \frac{1}{|z|^4} \right) = n \quad \dots(2)$$

Equation (1)² + (2)²

$$\left(1 + \frac{1}{|z|^4} \right)^2 [(x^2 + y^2)^2] = m^2 + n^2$$

$$\left(1 + \frac{1}{|z|^4} \right)^2 (|z|^4)^2 = m^2 + n^2$$

$$\Rightarrow |z|^4 + \frac{1}{|z|^4} + 2 = m^2 + n^2$$

Now for option (A)

$$|z|^4 = \frac{43 + 3\sqrt{205}}{2}$$

$$\Rightarrow m^2 + n^2 = 45$$

$$\Rightarrow m = \pm 6, n = \pm 3$$

Option (B)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{33}}{4} + \frac{7 - \sqrt{33}}{4} + 2 = \frac{7}{2} + 2 = \frac{11}{2}$$

Option (C)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{9 + \sqrt{65}}{4} + \frac{9 - \sqrt{65}}{4} + 2 = \frac{18}{4} + 2 = \frac{9}{2} + 2 = \frac{13}{2}$$

Option (D)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{13}}{6} + \frac{7 - \sqrt{13}}{6} + 2 = \frac{14}{6} + 2 = \frac{7}{3} + 2 = \frac{13}{2}$$

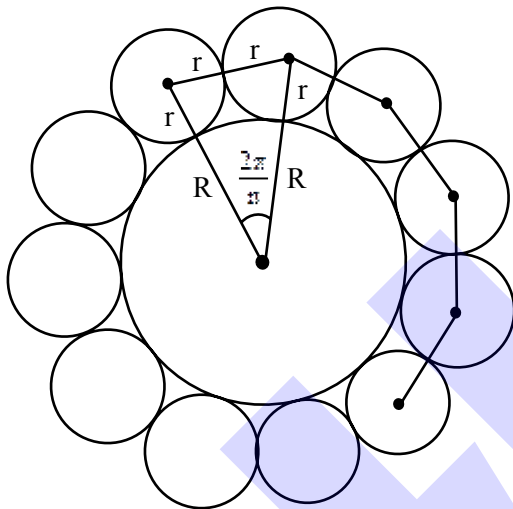
12. Let G be a circle of radius $R > 0$. Let G_1, G_2, \dots, G_n be n circles of equal radius $r > 0$. Suppose each of the n circles G_1, G_2, \dots, G_n touches the circle G externally. Also, for $i = 1, 2, \dots, n-1$, the circle G_i touches G_{i+1} externally, and G_n touches G_1 externally. Then, which of the following statements is/are TRUE ?

- (A) If $n = 4$, then $(\sqrt{2} - 1)r < R$
 (B) If $n = 5$, then $r < R$
 (C) If $n = 8$, then $(\sqrt{2} - 1)r < R$
 (D) If $n = 12$, then $\sqrt{2}(\sqrt{3} + 1)r > R$

Ans. (C,D)

Sol. $2(R+r)\sin\frac{\pi}{n} = 2r$

$$\frac{R+r}{r} = \operatorname{cosec} \frac{\pi}{n}$$



- (A) $n = 4, R + r = \sqrt{2}r$
 (B) $n = 5, \frac{R+r}{r} = \operatorname{cosec} \frac{\pi}{5} < \operatorname{cosec} \frac{\pi}{6}$
 $R + r < 2r \Rightarrow r > R$
 (C) $n = 8, \frac{R+r}{r} = \operatorname{cosec} \frac{\pi}{8} > \operatorname{cosec} \frac{\pi}{4}$
 $R + r > \sqrt{2}r$
 (D) $n = 12, \frac{R+r}{r} = \operatorname{cosec} \frac{\pi}{12} = \sqrt{2}(\sqrt{3} + 1)$
 $R + r = \sqrt{2}(\sqrt{3} + 1)r$
 $\sqrt{2}(\sqrt{3} + 1)r > R$

13. Let \hat{i}, \hat{j} and \hat{k} be the unit vectors along the three positive coordinate axes. Let

$$\vec{a} = 3\hat{i} + \hat{j} - \hat{k},$$

$$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}, \quad b_2, b_3 \in \mathbb{R},$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

be three vectors such that $b_2b_3 > 0$, $\vec{a} \cdot \vec{b} = 0$ and

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3 - c_1 \\ 1 - c_2 \\ -1 - c_3 \end{pmatrix}.$$

Then, which of the following is/are TRUE ?

(A) $\vec{a} \cdot \vec{c} = 0$

(B) $\vec{b} \cdot \vec{c} = 0$

(C) $|\vec{b}| > \sqrt{10}$

(D) $|\vec{c}| \leq \sqrt{11}$

Ans. (B,C,D)

Sol. $\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$

$$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3 - c_1 \\ 1 - c_2 \\ -1 - c_3 \end{pmatrix}$$

multiply & compare

$$b_2c_3 - b_3c_2 = c_1 - 3 \quad \dots(1)$$

$$c_3 - b_3c_1 = 1 - c_2 \quad \dots(2)$$

$$c_2 - b_2c_1 = 1 + c_3 \quad \dots(3)$$

$$(1)\hat{i} - (2)\hat{j} + (3)\hat{k}$$

$$\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(c_3 - b_3c_1) + \hat{k}(c_2 - b_2c_1)$$

$$= c_1\hat{i} + c_2\hat{j} + c_3\hat{k} - 3\hat{i} - \hat{j} + \hat{k}$$

$$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$$

Take dot product with \vec{b}

$$0 = \vec{c} \cdot \vec{b} - \vec{a} \cdot \vec{b}$$

$$\vec{b} \cdot \vec{c} = 0$$

$$\vec{b} \perp \vec{c}$$

$$\vec{b} \wedge \vec{c} = 90^\circ$$

Take dot product with \vec{c}

$$0 = |\vec{c}|^2 - \vec{a} \cdot \vec{c}$$

$$\vec{a} \cdot \vec{c} = |\vec{c}|^2$$

$$\vec{a} \cdot \vec{c} \neq 0$$

$$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$$

Squaring

$$|\vec{b}|^2 |\vec{c}|^2 = |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c} \cdot \vec{a}$$

$$|\vec{b}|^2 |\vec{c}|^2 = |\vec{c}|^2 + 11 - 2|\vec{c}|^2$$

$$|\vec{b}|^2 |\vec{c}|^2 = 11 - |\vec{c}|^2$$

$$|\vec{c}|^2 (|\vec{b}|^2 + 1) = 11$$

$$|\vec{c}|^2 = \frac{11}{|\vec{b}|^2 + 1}$$

$$|\vec{c}| \leq \sqrt{11}$$

given $\vec{a} \cdot \vec{b} = 0$

$$b_2 - b_3 = -3 \quad \text{also}$$

$$b_2^2 + b_3^2 - 2b_2b_3 = 9 \quad b_2b_3 > 0$$

$$b_2^2 + b_3^2 = 9 + 2b_2b_3$$

$$b_2^2 + b_3^2 = 9 + 2b_2b_3 > 9$$

$$b_2^2 + b_3^2 > 9$$

$$|\vec{b}| = \sqrt{1 + b_2^2 + b_3^2}$$

$$|\vec{b}| > \sqrt{10}$$

14. For $x \in \mathbb{R}$, let the function $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right), \quad y(0) = 0.$$

Then, which of the following statements is/are TRUE?

- (A) $y(x)$ is an increasing function
- (B) $y(x)$ is a decreasing function
- (C) There exists a real number β such that the line $y = \beta$ intersects the curve $y = y(x)$ at infinitely many points
- (D) $y(x)$ is a periodic function

Ans. (C)

Sol. $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right)$

Linear D.E.

$$\text{I.F.} = e^{\int 12 dx} = e^{12x}$$

Solution of DE

$$y \cdot e^{12x} = \int e^{12x} \cdot \cos\left(\frac{\pi}{12}x\right) dx$$

$$y \cdot e^{12x} = \frac{e^{12x}}{(12)^2 + \left(\frac{\pi}{12}\right)^2} \left(12 \cos\frac{\pi}{12}x + \frac{\pi}{12} \sin\frac{\pi}{12}x \right) + C$$

$$\Rightarrow y = \frac{(12)}{(12)^4 + \pi^2} \left((12)^2 \cos\left(\frac{\pi x}{12}\right) + \pi \sin\left(\frac{\pi x}{12}\right) \right) + \frac{C}{e^{12x}}$$

Given $y(0) = 0$

$$\Rightarrow 0 = \frac{12}{12^4 + \pi^2} (12^2 + 0) + C \Rightarrow C = \frac{-12^3}{12^4 + \pi^2}$$

$$\therefore y = \frac{12}{12^4 + \pi^2} \left[(12)^2 \cos\left(\frac{\pi x}{12}\right) + \pi \sin\left(\frac{\pi x}{12}\right) - 12^2 \cdot e^{-12x} \right]$$

$$\text{Now } \frac{dy}{dx} = \frac{12}{12^4 + \pi^2} \left[\underbrace{-12\pi \sin\left(\frac{\pi x}{12}\right) + \frac{\pi^2}{12} \cos\left(\frac{\pi x}{12}\right)}_{\text{min. value}} + 12^3 e^{-12x} \right]$$

$$\left(-\sqrt{144\pi^2 + \frac{\pi^4}{144}} = -12\pi \sqrt{1 + \frac{\pi^2}{12^4}} \right)$$

$$\Rightarrow \frac{dy}{dx} > 0 \quad \forall x \leq 0 \text{ \& \text{ may be negative/positive for } x > 0}$$

So, $f(x)$ is neither increasing nor decreasing

For some $\beta \in \mathbb{R}$, $y = \beta$ intersects $y = f(x)$ at infinitely many points

So option C is correct

SECTION-3 : (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +3 If **ONLY** the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : -1 In all other cases.

15. Consider 4 boxes, where each box contains 3 red balls and 2 blue balls. Assume that all 20 balls are distinct. In how many different ways can 10 balls be chosen from these 4 boxes so that from each box at least one red ball and one blue ball are chosen?

- (A) 21816 (B) 85536
 (C) 12096 (D) 156816

Ans. (A)

Sol.

3R 2B	3R 2B	3R 2B	3R 2B
B-1	B-2	B-3	B-4

Case-I : when exactly one box provides four balls (3R 1B or 2R 2B)

Number of ways in this case ${}^5C_4 ({}^3C_1 \times {}^2C_1)^3 \times 4$

Case-II : when exactly two boxes provide three balls (2R 1B or 1R 2B) each

Number of ways in this case $({}^5C_3 - 1)^2 ({}^3C_1 \times {}^2C_1)^2 \times 6$

Required number of ways = 21816

Language ambiguity : If we consider at least one red ball and exactly one blue ball, then required number of ways is 9504. None of the option is correct.

16. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then which of the following matrices is equal to M^{2022} ?

- (A) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$ (B) $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$
 (C) $\begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$ (D) $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

Ans. (A)

Sol. $M = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$$M = \begin{bmatrix} \frac{3}{2} + 1 & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} + 1 \end{bmatrix}$$

$$M = I + \frac{3}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M^{2022} = \left(I + \frac{3}{2} A \right)^{2022}$$

$$= I + 3033A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3033 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3034 & 3033 \\ -3033 & -3032 \end{bmatrix}$$

17. Suppose that

Box-I contains 8 red, 3 blue and 5 green balls,

Box-II contains 24 red, 9 blue and 15 green balls,

Box-III contains 1 blue, 12 green and 3 yellow balls,

Box-IV contains 10 green, 16 orange and 6 white balls.

A ball is chosen randomly from Box-I ; call this ball b . If b is red then a ball is chosen randomly from Box-II, if b is blue then a ball is chosen randomly from Box-III, and if b is green then a ball is chosen randomly from Box-IV. The conditional probability of the event 'one of the chosen balls is white' given that the event 'at least one of the chosen balls is green' has happened, is equal to

(A) $\frac{15}{256}$

(B) $\frac{3}{16}$

(C) $\frac{5}{52}$

(D) $\frac{1}{8}$

Ans. (C)

Sol. Box I 8(R) 3(B) 5(G)

Box II 24(R) 9(B) 15(G)

Box III 1(B) 12(G) 3(y)

Box IV 10(G) 16(o) 6(w)

A (one of the chosen balls is white)

B (at least one of the chosen ball is green)

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$A \cap B \rightarrow (wG)$

$$= \frac{\frac{5}{16} \times \frac{6}{32}}{\frac{5}{16} \times 1 + \frac{8}{16} \times \frac{15}{48} + \frac{3}{16} \times \frac{12}{16}}$$

$$= \frac{15}{156} = \frac{5}{52}$$

18. For positive integer n , define

$$f(n) = n + \frac{16 + 5n - 3n^2}{4n + 3n^2} + \frac{32 + n - 3n^2}{8n + 3n^2} + \frac{48 - 3n - 3n^2}{12n + 3n^2} + \dots + \frac{25n - 7n^2}{7n^2}.$$

Then, the value of $\lim_{n \rightarrow \infty} f(n)$ is equal to

(A) $3 + \frac{4}{3} \log_e 7$

(B) $4 - \frac{3}{4} \log_e \left(\frac{7}{3}\right)$

(C) $4 - \frac{4}{3} \log_e \left(\frac{7}{3}\right)$

(D) $3 + \frac{3}{4} \log_e 7$

Ans. (B)

$$18. \quad f(n) = n + \sum_{r=1}^n \frac{16r + (9-4r)n - 3n^2}{4rn + 3n^2}$$

$$f(n) = n + \sum_{r=1}^n \frac{(16r + 9n) - (4rn + 3n^2)}{4rn + 3n^2}$$

$$f(n) = n + \left(\sum_{r=1}^n \frac{16r + 9n}{4rn + 3n^2} \right) - n$$

$$\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{16r + 9n}{4rn + 3n^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\left(16 \left(\frac{r}{n} \right) + 9 \right) \frac{1}{n}}{4 \left(\frac{r}{n} \right) + 3}$$

$$= \int_0^1 \frac{16x + 9}{4x + 3} dx = \int_0^1 4 dx - \int_0^1 \frac{3 dx}{4x + 3}$$

$$= 4 - \frac{3}{4} (\ln |4x + 3|)_0^1$$

$$= 4 - \frac{3}{4} \ln \frac{7}{3}$$