

FINAL JEE(Advanced) EXAMINATION - 2021

(Held On Sunday 03rd OCTOBER, 2021)

PAPER-2

TEST PAPER WITH SOLUTION

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	: 0	If unanswered;
Negative Marks	: -2	In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
 choosing ONLY (A), (B) and (D) will get +4 marks;
 choosing ONLY (A) and (B) will get +2 marks;
 choosing ONLY (A) and (D) will get +2 marks;
 choosing ONLY (B) and (D) will get +2 marks;
 choosing ONLY (A) will get +1 mark;
 choosing ONLY (B) will get +1 mark;
 choosing ONLY (D) will get +1 mark;
 choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
 choosing any other option(s) will get -2 marks.

1. Let

$$S_1 = \{(i, j, k) : i, j, k \in \{1, 2, \dots, 10\}\}$$

$$S_2 = \{(i, j) : 1 \leq i < j + 2 \leq 10, i, j \in \{1, 2, \dots, 10\}\},$$

$$S_3 = \{(i, j, k, l) : 1 \leq i < j < k < l, i, j, k, l \in \{1, 2, \dots, 10\}\}.$$

and

$$S_4 = \{(i, j, k, l) : i, j, k \text{ and } l \text{ are distinct elements in } \{1, 2, \dots, 10\}\}.$$

If the total number of elements in the set S_r is n_r , $r = 1, 2, 3, 4$, then which of the following statements is (are) **TRUE**?

(A) $n_1 = 1000$

(B) $n_2 = 44$

(C) $n_3 = 220$

(D) $\frac{n_4}{12} = 420$

Ans. (A,B,D)

- Sol.** (A) $n_1 = 10 \times 10 \times 10 = 1000$
 (B) As per given condition $1 \leq i < j + 2 \leq 10 \Rightarrow j \leq 8 \text{ \& } i \geq 1$
 for $i = 1, 2, \quad j = 1, 2, 3, \dots, 8 \rightarrow (8 + 8)$ possibilities
 for $i = 3, \quad j = 2, 3, \dots, 8 \rightarrow 7$ possibilities
 $i = 4, \quad j = 3, \dots, 8 \rightarrow 6$ possibilities
 $i = 9, \quad j = 1 \rightarrow 1$ possibility
 So $n_2 = (1 + 2 + 3 + \dots + 8) + 8 = 44$
 (C) $n_3 = {}^{10}C_4$ (Choose any four)
 $= 210$
 (D) $n_4 = {}^{10}C_4 \cdot 4! = (210) (24)$
 $\Rightarrow \frac{n_4}{12} = 420$

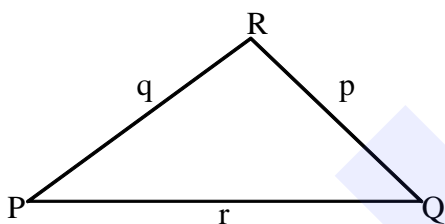
So correct Ans. (A), (B), (D)

2. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R , respectively. Then which of the following statements is (are) **TRUE**?

- (A) $\cos P \geq 1 - \frac{p^2}{2qr}$ (B) $\cos R \geq \left(\frac{q-r}{p+q}\right) \cos P + \left(\frac{p-r}{p+q}\right) \cos Q$
 (C) $\frac{q+r}{p} < 2 \frac{\sqrt{\sin Q \sin R}}{\sin P}$ (D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

Ans. (A,B)

Sol.



$$(A) \cos P = \frac{q^2 + r^2 - p^2}{2qr} = \frac{q^2 + r^2}{2qr} - \frac{p^2}{2qr} \geq 1 - \frac{p^2}{2qr}$$

(as $p^2 + q^2 \geq 2qr$ (AM $\geq$ GM)), so (A) is correct

$$(B) (p+q) \cos R \geq (q-r) \cos P + (p-r) \cos Q$$

$$\Rightarrow (p \cos R + r \cos P) + (q \cos R + r \cos Q) \geq q \cos P + p \cos Q$$

$$\Rightarrow q + p \geq r$$

So (B) is correct

$$(C) \frac{q+r}{p} = \frac{\sin Q + \sin R}{\sin P} \geq \frac{2\sqrt{\sin Q \sin R}}{\sin P} \text{ so (C) is incorrect}$$

$$(D) \cos Q > \frac{p}{r} \Rightarrow \sin R \cos Q > \sin P$$

$$\Rightarrow \sin P + \sin(R-Q) > 2 \sin P$$

$$\Rightarrow \sin(R-Q) > \sin P$$

need not necessarily hold true if $R < Q$

Hence (A), (B)

3. Let $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ be a continuous function such that

$$f(0) = 1 \text{ and } \int_0^{\frac{\pi}{3}} f(t) dt = 0$$

Then which of the following statements is (are) **TRUE**?

(A) The equation $f(x) - 3 \cos 3x = 0$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(B) The equation $f(x) - 3 \sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(C) $\lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{1 - e^{x^2}} = -1$

(D) $\lim_{x \rightarrow 0} \frac{\sin x \int_0^x f(t) dt}{x^2} = -1$

Ans. (A,B,C)

Sol. (A) Let $g(x) = f(x) - 3 \cos 3x$

$$\text{Now } \int_0^{\pi/3} g(x) dx = \int_0^{\pi/3} f(x) dx - 3 \int_0^{\pi/3} \cos 3x dx = 0$$

Hence $g(x) = 0$ has a root in $\left(0, \frac{\pi}{3}\right)$

(B) Let $h(x) = f(x) - 3 \sin 3x + \frac{6}{\pi}$

$$\begin{aligned} \text{Now } \int_0^{\pi/3} h(x) dx &= \int_0^{\pi/3} f(x) dx - 3 \int_0^{\pi/3} \sin 3x dx + \int_0^{\pi/3} \frac{6}{\pi} dx \\ &= 0 - 2 + 2 = 0 \end{aligned}$$

Hence $h(x) = 0$ has a root in $\left(0, \frac{\pi}{3}\right)$

$$\text{(C) } \lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{1 - e^{x^2}} = \lim_{x \rightarrow 0} \underbrace{\left(\frac{x^2}{1 - e^{x^2}} \right)}_{-1} \underbrace{\left(\frac{\int_0^x f(t) dt}{x} \right)}_{\text{Apply L' Hopital's Rule}}$$

$$= -1 \lim_{x \rightarrow 0} \frac{f(x)}{1} = -1$$

$$\begin{aligned} & (\sin x) \int_0^x f(t) dt \\ (D) \lim_{x \rightarrow 0} \frac{0}{x^2} \\ &= \lim_{x \rightarrow 0} \underbrace{\left(\frac{\sin x}{x} \right)}_1 \underbrace{\frac{\int_0^x f(t) dt}{x}}_{\text{Apply L' Hopitals Rule}} \\ &= 1 \lim_{x \rightarrow 0} \frac{f(x)}{1} = 1 \\ \text{Ans. A,B,C} \end{aligned}$$

4. For any real numbers α and β , let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx} + \alpha y = x e^{\beta x}, y(1) = 1$$

Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S ?

$$(A) f(x) = \frac{x^2}{2} e^{-x} + \left(e - \frac{1}{2}\right) e^{-x}$$

$$(B) f(x) = -\frac{x^2}{2} e^{-x} + \left(e + \frac{1}{2}\right) e^{-x}$$

$$(C) f(x) = \frac{e^x}{2} \left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right) e^{-x}$$

$$(D) f(x) = \frac{e^x}{2} \left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right) e^{-x}$$

Ans. (A,C)

Sol. Integrating factor = $e^{\alpha x}$

$$\text{So } y e^{\alpha x} = \int x e^{(\alpha+\beta)x} dx$$

Case-I

$$\text{If } \alpha + \beta = 0 \quad y e^{\alpha x} = \frac{x^2}{2} + c$$

$$\text{It passes through } (1, 1) \Rightarrow C = e^{\alpha} - \frac{1}{2}$$

$$\text{So } y e^{\alpha x} = \frac{x^2 - 1}{2} + e^{\alpha}$$

for $\alpha = 1$

$$y = \frac{x^2}{2} e^{-x} + \left(e - \frac{1}{2}\right) e^{-x} \rightarrow (A)$$

Case-II

If $\alpha + \beta \neq 0$

$$y e^{\alpha x} = \frac{x \cdot e^{(\alpha+\beta)x}}{\alpha + \beta} - \frac{1}{\alpha + \beta} e^{(\alpha+\beta)x} dx$$

$$\Rightarrow ye^{\alpha x} = \frac{xe^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + c$$

$$\Rightarrow \text{So } c = e^{\alpha} - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)^2} ((\alpha+\beta)x - 1) + e^{-\alpha x} \left(e^x - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2} \right)$$

If $\alpha = \beta = 1$

$$y = \frac{e^x}{4} (2x - 1) + e^{-x} \left(e - \frac{e^2}{2} + \frac{e^2}{4} \right)$$

$$y = \frac{e^x}{2} \left(x - \frac{1}{2} \right) + e^{-x} \left(e - \frac{e^2}{4} \right) \rightarrow (c)$$

Ans. (A) & (C)

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some

$\lambda > 0$. If $|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}$, then which of the following statements is (are) **TRUE**?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides

$\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Ans. (A,B,C)

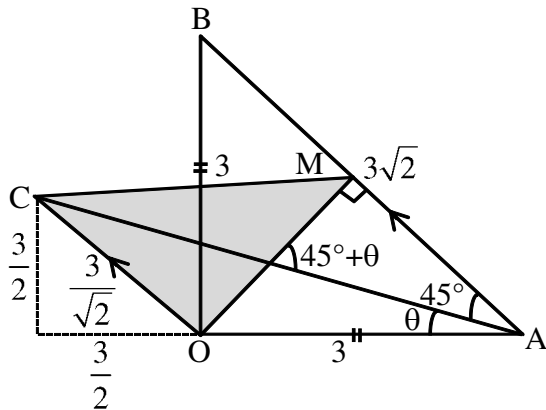
Sol. $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{1}{2} \overrightarrow{OB} \times (\overrightarrow{OB} - \lambda \overrightarrow{OA})$

$$= \frac{\lambda}{2} (\overrightarrow{OA} \times \overrightarrow{OB})$$

$$|\overrightarrow{OB}| \times |\overrightarrow{OC}| = \frac{|\lambda|}{2} |\overrightarrow{OA}| \times |\overrightarrow{OB}| \quad (\text{Note } \overrightarrow{OA} \text{ \& } \overrightarrow{OB} \text{ are perpendicular})$$

$$\Rightarrow \frac{9\lambda}{2} = \frac{9}{2} \Rightarrow \lambda = 1 \quad (\text{given } \lambda > 0)$$

$$\text{So } \overrightarrow{OC} = \frac{\overrightarrow{OB} - \overrightarrow{OA}}{2} = \frac{\overrightarrow{AB}}{2}$$



M is mid point of AB

Note projection of $\overline{OC}$ on $\overline{OA} = -\frac{3}{2}$

$$\tan \theta = \frac{1}{3}$$

$$\text{Area of } \triangle ABC = \frac{9}{2}$$

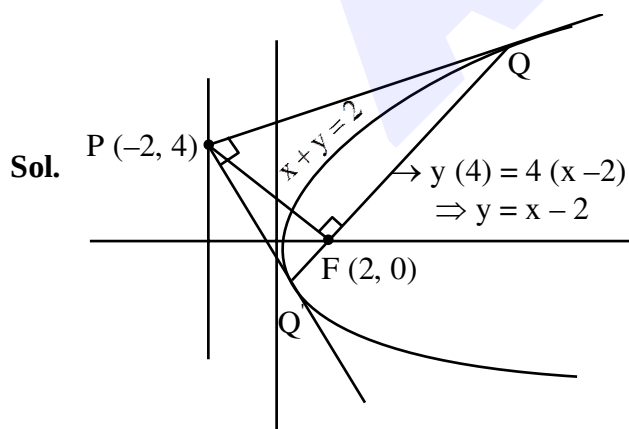
Acute angle between diagonals is

$$\tan^{-1} \left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} \right) = \tan^{-1} 2$$

6. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) **TRUE**?

- (A) The triangle PFQ is a right-angled triangle
(B) The triangle QPQ' is a right-angled triangle
(C) The distance between P and F is $5\sqrt{2}$
(D) F lies on the line joining Q and Q'

Ans. (A,B,D)



Note that P lies on directrix so triangle PQQ' is right angled, hence QQ' passes through focus F.

$$PF = 4\sqrt{2}$$

Equation of QF is $y = x - 2$ & PF is $x + y = 2$

Hence A, B, D.

SECTION-2 : (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

Zero Marks : 0 In all other cases.

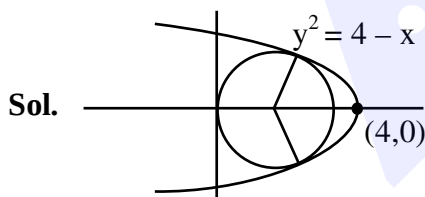
Question Stem for Questions Nos. 7 and 8

Question Stem

Consider the region $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x -axis. Let C be the circle that has largest radius among the circles in F . Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

7. The radius of the circle C is _____.

Ans. (1.50)



Let the circle be

$$x^2 + y^2 + \lambda x = 0$$

For point of intersection of circle & parabola $y^2 = 4 - x$.

$$x^2 + 4 - x + \lambda x = 0 \Rightarrow x^2 + x(\lambda - 1) + 4 = 0$$

For tangency : $\Delta = 0 \Rightarrow (\lambda - 1)^2 - 16 = 0 \Rightarrow \lambda = 5$ (rejected) or $\lambda = -3$

$$\text{Circle : } x^2 + y^2 - 3x = 0$$

$$\text{Radius} = \frac{3}{2} = 1.5$$

8. The value of α is _____.

Ans. (2.00)

Sol. For point of intersection :

$$x^2 - 4x + 4 = 0 \Rightarrow x = 2 \text{ so } \alpha = 2$$

Question Stem for Questions Nos. 9 and 10

Question Stem

Let $f_1 : (0, \infty) \rightarrow \mathbb{R}$ and $f_2 : (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j dt, \quad x > 0$$

and

$$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, \quad x > 0,$$

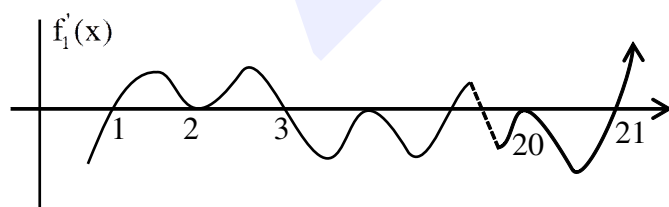
where, for any positive integer n and real numbers $a_1, a_2, \dots, a_n$, $\prod_{i=1}^n a_i$ denotes the product of $a_1, a_2, \dots, a_n$. Let m_i and n_i , respectively, denote the number of points of local minima and the number of points of local maxima of function f_i , $i = 1, 2$, in the interval $(0, \infty)$

9. The value of $2m_1 + 3n_1 + m_1n_1$ is _____.

Ans. (57.00)

Sol. $f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j dt$

$$f_1'(x) = \prod_{j=1}^{21} (x-j)^j = (x-1)(x-2)^2(x-3)^3 \dots (x-21)^{21}$$



So points of minima are $4m+1$ where $m = 0, 1, \dots, 5 \Rightarrow m_1 = 6$

Points of maxima are $4m-1$ where $m = 1, 2, \dots, 5 \Rightarrow n_1 = 5$

$$\Rightarrow 2m_1 + 3n_1 + m_1n_1 = 57$$

10. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____.

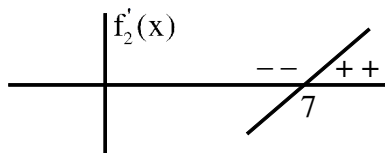
Ans. (6.00)

Sol. $f_2'(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$$\Rightarrow f_2'(x) = 2 \times 49 \times 50(x-1)^{49} - 50 \times 12 \times 49(x-1)^{48}$$

$$= 50 \times 49 \times 2(x-1)^{48}(x-1-6)$$

$$= 50 \times 49 \times 2(x-1)^{48}(x-7)$$



Point of minima = 7

$$\Rightarrow m_2 = 1$$

No point of maxima

$$\Rightarrow n_2 = 0$$

$$6m_2 + 4n_2 + 8m_2n_2 = 6$$

Question Stem for Questions Nos. 11 and 12

Question Stem

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \rightarrow \mathbb{R}$, $i = 1, 2$, and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \rightarrow \mathbb{R}$ be functions such that

$$g_1(x) = 1, g_2(x) = |4x - \pi| \text{ and } f(x) = \sin^2 x, \text{ for all } x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$$

$$\text{Define } S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x) dx, \quad i = 1, 2$$

11. The value of $\frac{16S_1}{\pi}$ is _____.

Ans. (2.00)

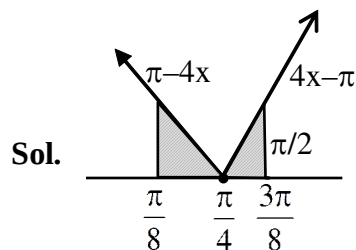
$$\text{Sol. } S_1 = \int_{\pi/8}^{3\pi/8} f(x) dx = \int_{\pi/8}^{3\pi/8} \sin^2 x dx = \int_{\pi/8}^{3\pi/8} \sin^2 \left(\frac{\pi}{8} + \frac{3\pi}{8} - x \right) dx = \int_{\pi/8}^{3\pi/8} \cos^2 x dx$$

$$2S_1 = \int_{\pi/8}^{3\pi/8} (\sin^2 x + \cos^2 x) dx = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$$

$$\Rightarrow \frac{16S_1}{\pi} = 2$$

12. The value of $\frac{48S_2}{\pi^2}$ is _____.

Ans. (1.50)



$$S_2 = \int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x |4x - \pi| dx$$

$$= \int_{\pi/8}^{3\pi/8} \sin^2 \left(\frac{\pi}{2} - x \right) \left| 4 \left(\frac{\pi}{2} - x \right) - \pi \right| dx$$

$$= \int_{\pi/8}^{3\pi/8} (\cos^2 x) |\pi - 4x| dx$$

$$\Rightarrow 2S_2 = \int_{\pi/8}^{3\pi/8} |4x - \pi| (\sin^2 x + \cos^2 x) dx = \int_{\pi/8}^{3\pi/8} |4x - \pi| dx$$

$$= 2 \times \frac{1}{2} \times \frac{\pi}{8} \times \frac{\pi}{2} = \frac{\pi^2}{16}$$

$$\Rightarrow \frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$$

SECTION-3 : (Maximum Marks : 12)

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +3	If ONLY the correct option is chosen;
Zero Marks	: 0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	: -1	In all other cases.

Paragraph

Let

$$M = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq r^2\},$$

where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, $n = 1, 2, 3, \dots$. Let $S_0 = 0$ and, for $n \geq 1$, let S_n denote the sum of the first n terms of this progression. For $n \geq 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n , and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n .

13. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M . Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

- (A) $k + 2l = 22$ (B) $2k + l = 26$ (C) $2k + 3l = 34$ (D) $3k + 2l = 40$

Ans. (D)

Sol. $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$
 $= 2 \left(1 - \frac{1}{2^n} \right) = 2 - \frac{1}{2^{n-1}}$

Centre of C_n is $\left(2 - \frac{1}{2^{n-2}}, 0 \right)$

and radius of C_n is $\frac{1}{2^{n-1}}$

when $r = \frac{1025}{513} < 2$

C_n will lie inside m

when $2 - \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} < \frac{1025}{513}$

$\Rightarrow k = 10$

Also $l = 5$

$3k + 2l = 30 + 10 = 40$

Ans. (D)

14. Consider M with $r = \frac{(2^{199} - 1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

- (A) 198 (B) 199 (C) 200 (D) 201

Ans. (B)

Sol. Center of D_n is (S_{n-1}, S_{n-1})

$$r = \frac{1}{2^{n-1}}$$

D_n will lie inside

$$\text{when } \sqrt{2}(S_{n-1}) < \frac{2^{199} - 1}{2^{198}} \sqrt{2}$$

$$\Rightarrow \frac{\sqrt{2}}{2^{n-2}} > \frac{\sqrt{2}}{2^{198}} + \frac{1}{2^{n-1}}$$

$$\Rightarrow n = 199$$

Paragraph

Let $\psi_1 : [0, \infty) \rightarrow \mathbb{R}$, $\psi_2 : [0, \infty) \rightarrow \mathbb{R}$, $f : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ be functions such that

$$f(0) = g(0) = 0,$$

$$\psi_1(x) = e^{-x} + x, \quad x \geq 0,$$

$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \quad x \geq 0,$$

$$f(x) = \int_{-x}^x (|t| - t^2) e^{-t^2} dt, \quad x > 0$$

and

$$g(x) = \int_0^{x^2} \sqrt{t} e^{-t} dt, \quad x > 0$$

15. Which of the following statements is **TRUE** ?

(A) $f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + \alpha x$

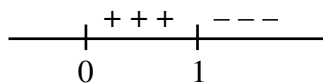
(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Ans. (C)

Sol. $f'(x) = (|x| - x^2)e^{-x^2} + (|x| - x^2)e^{-x^2}, x \geq 0$

$$f' = 2(x - x^2)e^{-x^2}$$



hence option (D) is wrong

$$g'(x) = xe^{-x^2} 2x$$

$$f'(x) + g'(x) = 2 \times e^{-x^2}$$

$$f(x) + g(x) = -e^{-x^2} + c$$

$$f(x) + g(x) = -e^{-x^2} + 1$$

$$F(\ln 3) + g(\sqrt{\ln 3}) = 1 - \frac{1}{3} = \frac{2}{3} \text{ (option (A) is wrong)}$$

$$H(x) = \psi_1(x) - 1 - \alpha x = e^{-x} + x - 1 - \alpha x, \quad x \geq 1 \text{ \& } \alpha \in (1, x)$$

$$H(1) = e^{-1} + 1 - 1 - \alpha < 0$$

$$H'(x) = -e^{-x} + 1 - \alpha > 0 \Rightarrow H(x) \text{ is } \downarrow \Rightarrow \text{option (B) is wrong}$$

$$(C) \quad \psi_2(x) = 2(\psi_1(\beta) - 1)$$

Applying L.M.V.T to $\psi_2(x)$ in $[0, x]$

$$\psi_2'(\beta) = \frac{\psi_2(x) - \psi_2(0)}{x}$$

$$2\beta - 2 + 2e^{-\beta} = \frac{\psi_2(x) - 0}{x}$$

$$\Rightarrow \psi_2(x) = 2x(\psi_1(\beta) - 1) \text{ has one solution}$$

option (C) is correct.

16. Which of the following statements is **TRUE** ?

(A) $\psi_1(x) \leq 1$, for all $x > 0$

(B) $\psi_2(x) \leq 0$, for all $x > 0$

(C) $f(x) \geq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Ans. (D)

Sol. (A) $\psi_1(x) = e^{-x} + x, x \geq 0$

$$\psi_1'(x) = 1 - e^{-x} > 0 \Rightarrow \psi_1(x) \text{ is } \uparrow$$

$$\psi_1(x) \geq \psi_1(0) \quad \forall x \geq 0 \Rightarrow \psi_1(x) \geq 1$$

(B) $\psi_2(x) = x^2 - 2x + 2 - 2e^{-x} \quad x \geq 0$

$$\psi_2'(x) = 2x - 2 + 2e^{-x} = 2\psi_1(x) - 2 \geq 0 \quad \forall x \geq 0$$

$$\Rightarrow \psi_2(x) \text{ is } \uparrow \Rightarrow \psi_2(x) \geq \psi_2(0) \Rightarrow \psi_2(x) \geq 0$$

$$(C) f(x) = 2 \int_0^x (t - t^2) e^{-t^2} dt \quad \& \quad x \in \left(0, \frac{1}{2}\right)$$

$$= \int_0^x 2te^{-t^2} dt - \int_0^x 2t^2 e^{-t^2} dt$$

$$= -e^{-x^2} \Big|_0^x -$$

$$\text{Let } H(x) = f(x) - 1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5, \quad x \in \left(0, \frac{1}{2}\right)$$

$$H(0) = 0$$

$$H'(x) = 2(x - x^2)e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$$

$$= -2x^2e^{-x^2} + 2x^2 - 2x^4$$

$$= 2x^2(1 - x^2 - e^{-x^2})$$

$$\therefore e^{-x^2} \geq 1 - x^2 \quad \forall x \geq 0$$

$$\Rightarrow H'(x) \leq 0$$

$$\Rightarrow H(x) \text{ is } \downarrow \Rightarrow 1 - 1(x) < 0 \quad \forall x \in \left(0, \frac{1}{2}\right)$$

$$\text{Let } P(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7, \quad x \in \left(0, \frac{1}{2}\right)$$

$$P'(x) = 2x^2e^{-x^2} - 2x^2 + 2x^4 - x^6$$

$$= 2x^2 \left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{3} + \dots \right) - 2x^2 + 2x^4 - x^6$$

$$= -\frac{x^8}{3} + \frac{x^{10}}{12} \dots\dots\dots$$

$$\Rightarrow P'(x) \leq 0$$

$$\Rightarrow P(x) \text{ is } \downarrow$$

$$\Rightarrow P(x) \leq 0$$

option (D) is correct

SECTION-4 : (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If **ONLY** the correct integer is entered;

Zero Marks : 0 In all other cases.

17. A number is chosen at random from the set $\{1, 2, 3, \dots, 2000\}$. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is ____.

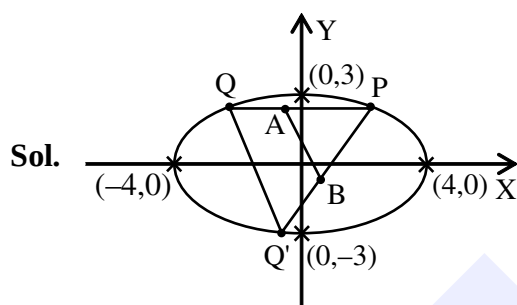
Ans. (214)

Sol. A = set of numbers divisible by 3
 $A = \{3, 6, 9, 12, \dots, 1998\}$
 $\therefore n(A) = 666$
 B = set of numbers divisible by 7
 $B = \{7, 14, 21, \dots, 1995\}$
 $\therefore n(B) = 285$
 $A \cap B = \{21, 42, \dots, 1995\}$
 $\therefore n(A \cup B) = 606 + 285 - 95 = 856$
 required probability = $\frac{856}{2000} = P$

$$\text{so, } 500P = \frac{856}{2000} \times 500 = 214$$

- 18.** Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P, Q and Q' on E, let M(P, Q) be the mid-point of the line segment joining P and Q, and M(P, Q') be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between M(P, Q) and M(P, Q'), as P, Q and Q' vary on E, is ____.

Ans. (4)



A and B be midpoints of segment PQ and PQ' respectively

$$AB = \text{distance between } M(P, Q) \text{ and } M(P, Q') = \frac{1}{2} \cdot QQ'$$

Since, Q, Q' must be on E, so,
 maximum of $QQ' = 8$

$$\therefore \text{Maximum of } AB = \frac{8}{2} = 4$$

- 19.** For any real number x, let $[x]$ denote the largest integer less than or equal to x. If

$$I = \int_0^{10} \left\lfloor \sqrt{\frac{10x}{x+1}} \right\rfloor dx,$$

then the value of $9I$ is ____.

Ans. (182)

Sol. Let $f(x) = \left\lfloor \frac{10x}{x+1} \right\rfloor$

$$\text{So, } f'(x) = 10 \left(\frac{(x+1) - x}{(x+1)^2} \right) = \frac{10}{(x+1)^2} > 0 \quad \forall x \in [0, 10],$$

So, $f(x)$ is an increasing function

So, range of $f(x)$ is $\left[0, \sqrt{\frac{100}{11}}\right]$

$$\begin{aligned} I &= \int_0^{1/9} \left[\sqrt{\frac{10x}{x+1}} \right] dx + \int_{2/3}^9 \left[\sqrt{\frac{10x}{x+1}} \right] dx + \int_{1/9}^{2/3} \left[\sqrt{\frac{10x}{x+1}} \right] dx + \int_{2/3}^9 \left[\sqrt{\frac{10x}{x+1}} \right] dx + \int_9^{10} \left[\sqrt{\frac{10x}{x+1}} \right] dx \\ &= 0 + \int_{1/9}^{2/3} dx + 2 \int_{2/3}^9 dx + 3 \int_9^{10} dx \\ &= \frac{2}{3} - \frac{1}{9} + 2 \left(9 - \frac{2}{3} \right) + 3(10 - 9) \\ &= \frac{6-1}{9} + 2 \times \frac{25}{3} + 3 = \frac{5}{9} + \frac{50}{3} + 3 \\ &= \frac{5+150+27}{9} = \frac{182}{9} = 182 \end{aligned}$$

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