

FINAL JEE(Advanced) EXAMINATION - 2021

(Held On Sunday 03rd OCTOBER, 2021)

PAPER-1

TEST PAPER WITH SOLUTION

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : -1 In all other cases.

1. Consider a triangle Δ whose two sides lie on the x-axis and the line $x + y + 1 = 0$. If the orthocenter of Δ is $(1, 1)$, then the equation of the circle passing through the vertices of the triangle Δ is

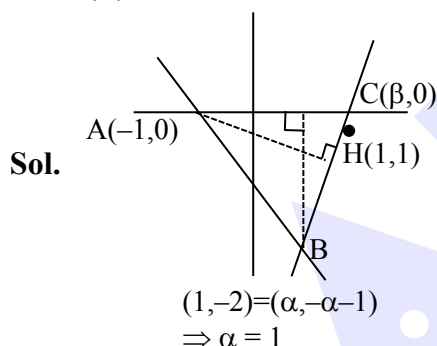
(A) $x^2 + y^2 - 3x + y = 0$

(B) $x^2 + y^2 + x + 3y = 0$

(C) $x^2 + y^2 + 2y - 1 = 0$

(D) $x^2 + y^2 + x + y = 0$

Ans. (B)



one of the vertex is intersection of x-axis and $x + y + 1 = 0 \Rightarrow A(-1, 0)$

Let vertex B be $(\alpha, -\alpha - 1)$

Line $AC \perp BH \Rightarrow \alpha = 1 \Rightarrow B(1, -2)$

Let vertex C be $(\beta, 0)$

Line $AH \perp BC$

$$m_{AH} \cdot m_{BC} = -1$$

$$\frac{1}{2} \cdot \frac{2}{\beta - 1} = -1 \Rightarrow \beta = 0$$

Centroid of ΔABC is $\left(0, -\frac{2}{3}\right)$

Now G(centroid) divides line joining circum centre (O) and ortho centre (H) in the ratio 1 : 2

$$\Rightarrow \begin{array}{c} (h,k) \quad \left(0, -\frac{2}{3}\right) \quad (1,1) \\ \text{O} \quad 1 \quad \text{G} \quad 2 \quad \text{H} \end{array}$$

$$2h + 1 = 0 \quad 2k + 1 = -z$$

$$h = -\frac{1}{2} \quad k = -\frac{3}{2}$$

$$\Rightarrow \text{circum centre is } \left(-\frac{1}{2}, -\frac{3}{2}\right)$$

Equation of circum circle is (passing through C(0,0)) is

$$x^2 + y^2 + x + 3y = 0$$

2. The area of the region $\{(x, y) : 0 \leq x \leq \frac{9}{4}, 0 \leq y \leq 1, x \geq 3y, x + y \geq 2\}$ is

(A) $\frac{11}{32}$

(B) $\frac{35}{96}$

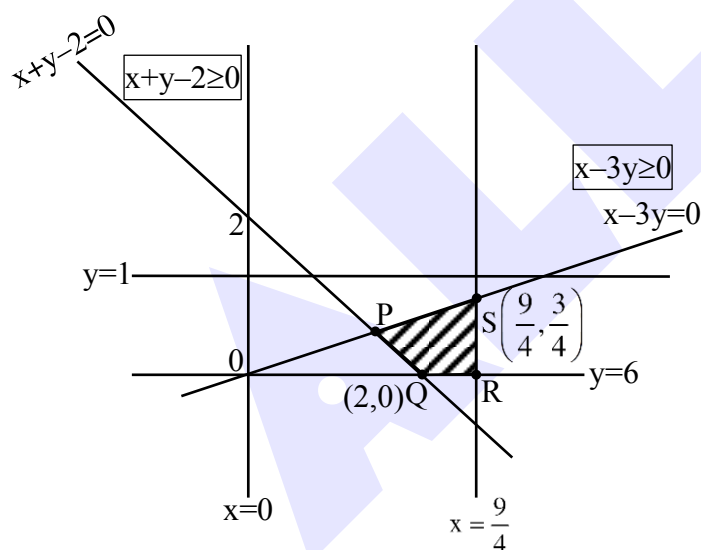
(C) $\frac{37}{96}$

(D) $\frac{13}{32}$

Ans. (A)

Sol. $x + y - 2 = 0$

$$P\left(\frac{3}{2}, \frac{1}{2}\right); Q(2, 0); R\left(\frac{9}{4}, 0\right); S\left(\frac{9}{4}, \frac{3}{4}\right)$$



$$\text{Area} = \frac{1}{2} \left| \begin{vmatrix} \frac{3}{2} & \frac{1}{2} \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ \frac{9}{4} & 0 \end{vmatrix} + \begin{vmatrix} \frac{9}{4} & 0 \\ \frac{9}{4} & \frac{3}{4} \end{vmatrix} + \begin{vmatrix} \frac{9}{4} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| (0-1) + (0-0) + \left(\frac{27}{16} - 0\right) + \left(\frac{9}{8} - \frac{9}{8}\right) \right| = \frac{11}{32}$$

3. Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1 , and let S_1 denote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 denote the set of these chosen elements. Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement, from the set G_2 and let S_3 denote the set of these chosen elements. Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is

- (A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{2}{5}$

Ans. (A)

Sol.
$$P = \frac{P(S_1 \cap (E_1 = E_3))}{P(E_1 = E_3)} = \frac{P(B_{1,2})}{P(B)}$$

$$P(B) = P(B_{1,2}) + P(B_{1,3}) + P(B_{2,3})$$

$\uparrow \quad \quad \uparrow \quad \quad \uparrow$
 If 1,2 If 1,3 If 2,3
 chosen chosen chosen
 at start at start at start

$$P(B_{1,2}) = \frac{1}{3} \times \frac{1 \times {}^3C_1}{{}^4C_2} \times \frac{1}{{}^5C_2}$$

$\underbrace{\hspace{1.5cm}}$ $\underbrace{\hspace{1.5cm}}$
 1 is definitely 1,2 chosen
 chosen from F_2 from G_2

$$P(B_{1,3}) = \frac{1}{3} \times \frac{1 \times {}^2C_1}{{}^3C_2} \times \frac{1}{{}^5C_2}$$

$\underbrace{\hspace{1.5cm}}$ $\underbrace{\hspace{1.5cm}}$
 1 is definitely 1,2 chosen
 chosen from F_2 from G_2

$$P(B_{2,3}) = \frac{1}{3} \times \left[\frac{{}^3C_2 \times 1}{{}^4C_2} \times \frac{1}{{}^4C_2} + \frac{1 \times {}^3C_1}{{}^4C_2} \times \frac{1}{{}^5C_2} \right]$$

$\underbrace{\hspace{1.5cm}}$ $\underbrace{\hspace{1.5cm}}$
 If 1 is not chosen If 1 is chosen
 from F_2 from F_2

$$\frac{P(B_{1,2})}{P(B)} = \frac{1}{5}$$

4. Let $\theta_1, \theta_2, \dots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \dots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, \dots, 10$, where $i = \sqrt{-1}$. Consider the statements P and Q given below :

P : $|z_2 - z_1| + |z_3 - z_2| + \dots + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$

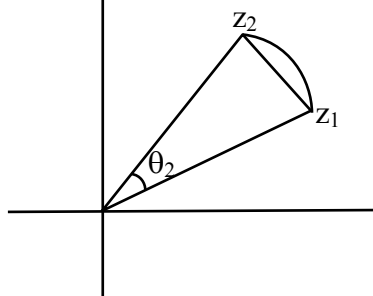
Q : $|z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \dots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$

Then,

- (A) P is **TRUE** and Q is **FALSE** (B) Q is **TRUE** and P is **FALSE**
 (C) both P and Q are **TRUE** (D) both P and Q are **FALSE**

Ans. (C)

Sol.



$$|z_1| = |z_2| = \dots |z_{10}| = 1$$

$$\text{angle} = \frac{\text{arc}}{\text{rad}}$$

$$\theta_2 = \text{arc}(z_1 z_2) > (z_2 > z_1)$$

$$P : |z_2 - z_1| + \dots + |z_1 - z_{10}| \leq \theta_1 + \theta_2 + \dots + \theta_{10}$$

$$\Rightarrow |z_2 - z_1| + \dots + |z_1 - z_{10}| \leq 2\pi \text{ P is true}$$

$$z_1^2 = e^{i2\theta_1}, z_k^2 = z_{k-1}^2 \cdot e^{i2\theta_k}$$

$$\text{Let } 2\theta_k = \alpha_k$$

$$z_1^2 = e^{i\alpha_1}, z_k^2 = z_{k-1}^2 \cdot e^{i\alpha_k}$$

$$\alpha_1 + \alpha_2 + \dots + \alpha_k = 4\pi$$

one similar sense

$$|z_1^2 - z_2^2| + \dots + |z_1^2 - z_{10}^2| \leq 4\pi$$

Q is also true

SECTION-2 : (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +2	If ONLY the correct numerical value is entered at the designated place;
Zero Marks	: 0	In all other cases.

Question Stem for Question Nos. 5 and 6

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set $S = \{1, 2, 3, \dots, 100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

5. The value of $\frac{625}{4}p_1$ is _____.

Ans. (76.25)

Sol. p_1 = probability that maximum of chosen numbers is at least 81

$p_1 = 1 - \text{probability that maximum of chosen number is at most 80}$

$$p_1 = 1 - \frac{80 \times 80 \times 80}{100 \times 100 \times 100} = 1 - \frac{64}{125}$$

$$p_1 = \frac{61}{125}$$

$$\frac{625p_1}{4} = \frac{625}{4} \times \frac{61}{125} = \frac{305}{4} = 76.25$$

the value of $\frac{625p_1}{4}$ is 76.25

6. The value of $\frac{125}{4}p_2$ is _____.

Ans. (24.50)

Sol. p_2 = probability that minimum of chosen numbers is at most 40

$= 1 - \text{probability that minimum of chosen numbers is at least 41}$

$$= 1 - \left(\frac{60}{100}\right)^3$$

$$= 1 - \frac{27}{125} = \frac{98}{125}$$

$$\therefore \frac{125}{4}p_2 = \frac{125}{4} \times \frac{98}{125} = 24.50$$

Question Stem for Question Nos. 7 and 8
Question Stem

Let α , β and γ be real numbers such that the system of linear equations

$$x + 2y + 3z = \alpha$$

$$4x + 5y + 6z = \beta$$

$$7x + 8y + 9z = \gamma - 1$$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point $(0, 1, 0)$ from the plane P .

7. The value of $|M|$ is _____.

Ans. (1.00)

8. The value of D is _____.

Ans. (1.50)

Solutions 7 & 8

Sol. $7x + 8y + 9z - (\gamma - 1) = A(4x + 5y + 6z - \beta) + B(x + 2y + 3z - \alpha)$

$$x : 7 = 4A + B$$

$$y : 8 = 5A + 2B$$

$$A = 2, B = -1$$

$$\text{const. term : } -(\gamma - 1) = -A\beta - \alpha B \Rightarrow -(\gamma - 1) = 2\beta + \alpha$$

$$\alpha - 2\beta + \gamma = 1$$

$$M = \begin{pmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \alpha - 2\beta + \gamma = 1$$

$$\text{Plane } P : x - 2y + z = 1$$

$$\text{Perpendicular distance} = \left| \frac{3}{\sqrt{6}} \right| = P \Rightarrow D = P^2 = \frac{9}{6} = 1.5$$

Question Stem for Question Nos. 9 and 10

Question Stem

Consider the lines L_1 and L_2 defined by

$$L_1 : x\sqrt{2} + y - 1 = 0 \text{ and } L_2 : x\sqrt{2} - y + 1 = 0$$

For a fixed constant λ , let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2 . The line $y = 2x + 1$ meets C at two points R and S , where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S' . Let D be the **square** of the distance between R' and S' .

9. The value of λ^2 is _____.

Ans. (9.00)

Sol. $P(x, y) \quad \left| \frac{\sqrt{2}x + y - 1}{\sqrt{3}} \right| \left| \frac{\sqrt{2}x - y + 1}{\sqrt{3}} \right| = \lambda^2$

$$\left| \frac{2x^2 - (y-1)^2}{3} \right| = \lambda^2, \quad C: |2x^2 - (y-1)^2| = 3\lambda^2$$

line $y = 2x + 1$, $RS = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, $R(x_1, y_1)$ and $S(x_2, y_2)$

$y_1 = 2x_1 + 1$ and $y_2 = 2x_2 + 1 \Rightarrow (y_1 - y_2) = 2(x_1 - x_2)$

$RS = \sqrt{5(x_1 - x_2)^2} = \sqrt{5}|x_1 - x_2|$

solve curve C and line $y = 2x + 1$ we get

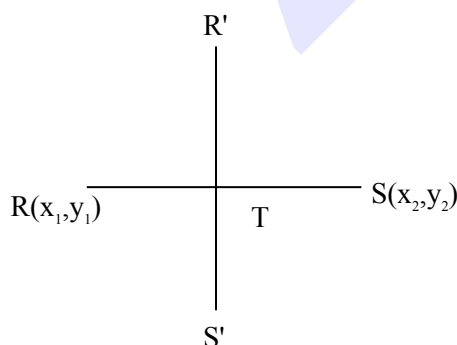
$|2x^2 - (2x)^2| = 3\lambda^2 \Rightarrow x^2 = \frac{3\lambda^2}{2}$

$RS = \sqrt{5} \left| \frac{2\sqrt{3}\lambda}{\sqrt{2}} \right| = \sqrt{30}\lambda = \sqrt{270} \Rightarrow 30\lambda^2 = 270 \Rightarrow \lambda^2 = 9$

10. The value of D is _____.

Ans. (77.14)

Sol.



⊥ bisector of RS

$$T \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Here $x_1 + x_2 = 0$

$$T = (0, 1)$$

Equation of

$$R'S' : (y - 1) = -\frac{1}{2}(x - 0) \Rightarrow x + 2y = 2$$

$$R'(a_1, b_1) \quad S'(a_2, b_2)$$

$$D = (a_1 - a_2)^2 + (b_1 - b_2)^2 = 5(b_1 - b_2)^2$$

$$\text{solve } x + 2y = 2 \text{ and } |2x^2 - (y - 1)^2| = 3\lambda^2$$

$$|8(y - 1)^2 - (y - 1)^2| = 3\lambda^2 \Rightarrow (y - 1)^2 = \left(\frac{\sqrt{3}\lambda}{\sqrt{7}} \right)^2$$

$$y - 1 = \pm \frac{\sqrt{3}\lambda}{\sqrt{7}} \Rightarrow b_1 = 1 + \frac{\sqrt{3}\lambda}{\sqrt{7}}, b_2 = 1 - \frac{\sqrt{3}\lambda}{\sqrt{7}}$$

$$D = 5 \left(\frac{2\sqrt{3}\lambda}{\sqrt{7}} \right)^2 = \frac{5 \times 4 \times 3\lambda^2}{7} = \frac{5 \times 4 \times 27}{7} = 77.14$$

SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all the four options are correct but **ONLY** three options are chosen;

Partial Marks : +2 If three or more options are correct but **ONLY** two options are chosen, both of which are correct;

Partial Marks : +1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option;

Zero Marks : 0 If unanswered;

Negative Marks : -2 In all other cases.

11. For any 3×3 matrix M , let $|M|$ denote the determinant of M . Let

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3×3 , then which of the following statements is (are) **TRUE** ?

(A) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $|(EF)^3| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

Ans. (A,B,D)

$$PEP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

$|E| = 0$ and $|F| = 0$ and $|Q| \neq 0$

$|EQ| = |E||Q| = 0$, $|PFQ^{-1}| = \frac{|P||F|}{|Q|} = 0$

$T = EQ + PFQ^{-1}$

$TQ = EQ^2 + PF = EQ^2 + P^2EP = EQ^2 + EP = E(Q^2 + P)$

$|TQ| = |E(Q^2 + P)| \Rightarrow |T||Q| = |E||Q^2 + P| = 0 \Rightarrow |T| = 0$ (as $|Q| \neq 0$)

(C) $|(EF)^3| > |EF|^2$

Here $0 > 0$ (false)

(D) as $P^2 = I \Rightarrow P^{-1} = P$ so $P^{-1}FP = PFP = PPEPP = E$

so $E + P^{-1}FP = E + E = 2E$

$P^{-1}EP + F \Rightarrow PEP + F = 2PEP$

$$\text{Tr}(2\text{PEP}) = 2\text{Tr}(\text{PEP}) = 2\text{Tr}(\text{EPP}) = 2\text{Tr}(\text{E})$$

12. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}.$$

Then which of the following statements is (are) **TRUE** ?

(A) f is decreasing in the interval $(-2, -1)$

(B) f is increasing in the interval $(1, 2)$

(C) f is onto

(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

Ans. (A,B)

Sol. $f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$$f'(x) = \frac{(x^2 + 2x + 4)(2x - 3) - (x^2 - 3x - 6)(2x + 2)}{(x^2 + 2x + 4)^2}$$

$$f'(x) = \frac{5x(x + 4)}{(x^2 + 2x + 4)^2}$$

$$f'(x): \begin{array}{c} + \quad - \quad + \\ \hline -4 \quad \quad 0 \end{array}$$

$$f(-4) = \frac{11}{6}, \quad f(0) = -\frac{3}{2}, \quad \lim_{x \rightarrow \pm\infty} f(x) = 1$$

Range: $\left[-\frac{3}{2}, \frac{11}{6}\right]$, clearly $f(x)$ is into

13. Let E, F and G be three events having probabilities

$$P(E) = \frac{1}{8}, P(F) = \frac{1}{6} \text{ and } P(G) = \frac{1}{4}, \text{ and let } P(E \cap F \cap G) = \frac{1}{10}.$$

For any event H, if H^c denotes its complement, then which of the following statements is(are) **TRUE** ?

(A) $P(E \cap F \cap G^c) \leq \frac{1}{40}$

(B) $P(E^c \cap F \cap G) \leq \frac{1}{15}$

(C) $P(E \cup F \cup G) \leq \frac{13}{24}$

(D) $P(E^c \cap F^c \cap G^c) \leq \frac{5}{12}$

Ans. (A,B,C)

Sol. $P(E) = \frac{1}{8}; P(F) = \frac{1}{6}; P(G) = \frac{1}{4}; P(E \cap F \cap G) = \frac{1}{10}$

$$\begin{aligned} \text{(C)} P(E \cup F \cup G) &= P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(G \cap E) + P(E \cap F \cap G) \\ &= \frac{1}{8} + \frac{1}{6} + \frac{1}{4} - \sum P(E \cap F) + \frac{1}{10} \end{aligned}$$

$$= \frac{3+4+6}{24} + \frac{1}{10} - \sum P(E \cap F) = \frac{13}{24} + \frac{1}{10} - \sum P(E \cap F)$$

$$\Rightarrow P(E \cup F \cup G) \leq \frac{13}{24} \quad [(C) \text{ is Correct}]$$

$$(D) P(E^C \cap F^C \cap G^C) = 1 - P(E \cup F \cup G) \geq 1 - \frac{13}{24}$$

$$\Rightarrow P(E^C \cap F^C \cap G^C) \geq \frac{11}{24} \quad [(D) \text{ is Incorrect}]$$

$$(A) P(E) = \frac{1}{8} \geq P(E \cap F \cap G^C) + P(E \cap F \cap G)$$

$$\Rightarrow \frac{1}{8} \geq P(E \cap F \cap G^C) + \frac{1}{10} \Rightarrow \frac{1}{8} - \frac{1}{10} \geq P(E \cap F \cap G^C)$$

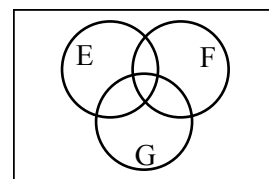
$$\Rightarrow \frac{1}{40} \geq P(E \cap F \cap G^C) \quad [(A) \text{ is Correct}]$$

$$(B) P(F) = \frac{1}{6} \geq P(E^C \cap F \cap G) + P(E \cap F \cap G)$$

$$\Rightarrow \frac{1}{6} - \frac{1}{10} \geq P(E^C \cap F \cap G)$$

$$\Rightarrow \frac{4}{60} \geq P(E^C \cap F \cap G)$$

$$\Rightarrow \frac{1}{15} \geq P(E^C \cap F \cap G) \quad [(B) \text{ is Correct}]$$



14. For any 3×3 matrix M , let $|M|$ denote the determinant of M . Let I be the 3×3 identity matrix. Let E and F be two 3×3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE** ?

$$(A) |FE| = |I - FE| |FGE|$$

$$(B) |I - FE| (I + FGE) = I$$

$$(C) EFG = GEF$$

$$(D) (I - FE)(I - FGE) = I$$

Ans. (A,B,C)

Sol. $|I - EF| \neq 0$; $G = (I - EF)^{-1} \Rightarrow G^{-1} = I - EF$

Now, $G \cdot G^{-1} = I = G^{-1} \cdot G$

$$\Rightarrow G(I - EF) = I = (I - EF)G$$

$$\Rightarrow G - GEF = I = G - EFG$$

$$\Rightarrow GEF = EFG \quad [C \text{ is Correct}]$$

$$\begin{aligned} (I - FE)(I + FGE) &= I + FGE - FE - FEFGE \\ &= I + FGE - FE - F(G - I)E \\ &= I + FGE - FE - FGE + FE \\ &= I \quad [(B) \text{ is Correct}] \end{aligned}$$

(So 'D' is Incorrect)

We have

$$(I - FE)(I + FGE) = I \quad \dots (I)$$

Now

$$\begin{aligned}
 & FE(I + FGE) \\
 &= FE + FEFGE \\
 &= FE + F(G - I)E \\
 &= FE + FGE - FE \\
 &= FGE \\
 &\Rightarrow |FE| |I + FGE| = |FGE| \\
 &\Rightarrow |FE| \times \frac{1}{|I - FE|} = |FGE| \text{ (from (1))} \\
 &\Rightarrow |FE| = |I - FE| |FGE| \\
 &\text{(option (A) is correct)}
 \end{aligned}$$

15. For any positive integer n , let $S_n : (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^n \cot^{-1} \left(\frac{1 + k(k+1)x^2}{x} \right),$$

where for any $x \in \mathbb{R}$, $\cot^{-1} x \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following

statements is (are) **TRUE** ?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1} \left(\frac{1 + 11x^2}{10x} \right)$, for all $x > 0$

(B) $\lim_{n \rightarrow \infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(D) $\tan(S_n(x)) \leq \frac{1}{2}$, for all $n \geq 1$ and $x > 0$

Ans. (A,B)

Sol.

$$\begin{aligned}
 S_n(x) &= \sum_{k=1}^n \tan^{-1} \left(\frac{x}{1 + kx(kx + x)} \right) \\
 &= \sum_{k=1}^n \tan^{-1} \left(\frac{(kx + x) - (kx)}{1 + (kx + x)(kx)} \right) \\
 S_n(x) &= \tan^{-1}(nx + x) - \tan^{-1}x = \tan^{-1} \left(\frac{nx}{1 + (n+1)x^2} \right) \\
 \text{(A) } S_{10}(x) &= \tan^{-1} \frac{10x}{1 + 11x^2} = \frac{\pi}{2} - \tan^{-1} \left(\frac{1 + 11x^2}{10x} \right) \quad (x > 0)
 \end{aligned}$$

$$(B) \lim_{n \rightarrow \infty} \cot(S_n(x)) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \left(1 + \frac{1}{n}\right)x^2}{x} = x \quad (x > 0)$$

$$(C) S_3(x) = \tan^{-1} \frac{3x}{1+4x^2} = \frac{\pi}{4} \Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin \mathbb{R}$$

$$(D) \tan(S_n(x)) = \frac{nx}{1+(n+1)x^2} ; \forall n \geq 1 ; x > 0$$

We need to check the validity of $\frac{nx}{1+(n+1)x^2} \leq \frac{1}{2} \quad \forall n \geq 1 ; x > 0 ; n \in \mathbb{N}$

$$\Rightarrow 2nx \leq (n+1)x^2 + 1$$

$$\Rightarrow (n+1)x^2 - 2nx + 1 \geq 0 \quad \forall n \geq 1 ; x > 0 ; n \in \mathbb{N}$$

Discriminant of $y = (n+1)x^2 - 2nx + 1$ is

$$D = 4n^2 - 4(n+1) \text{ and } n \in \mathbb{N}$$

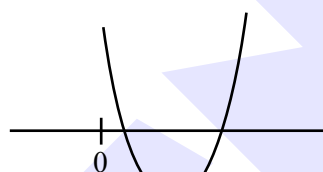
$D < 0$ for $n = 1$; true for $x > 0$

$D > 0$ for $n \geq 2 \Rightarrow \exists$ some $x > 0$

for which $y < 0$ as both roots of $y = 0$ will be positive.

$$y = (n+1)x^2 - 2nx + 1, n \geq 2$$

So, $y \geq 0 \quad \forall n \geq 1 ; \forall x > 0 ; n \in \mathbb{N}$ is false.



16. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0.$$

Then which of the following statements is (are) **TRUE** ?

(A) $\alpha = -1$

(B) $\alpha\beta = 4$

(C) $\alpha\beta = -4$

(D) $\beta = 4$

Ans. (B,D)

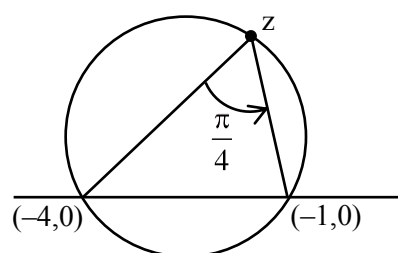
Sol. $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$ implies z is

on arc and $(-\alpha, 0)$ & $(-\beta, 0)$ subtend $\frac{\pi}{4}$ on z .

And z lies on $x^2 + y^2 + 5x - 3y + 4 = 0$

So put $y = 0$;

$$x^2 + 5x + 4 = 0 \Rightarrow x = -1 ; x = -4$$



Now, $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4} \Rightarrow z+\alpha = (z+\beta) \cdot r \cdot e^{i\frac{\pi}{4}}$

So, $z+\beta = z+4 \Rightarrow \beta = 4$ & $z+\alpha = z+1 \Rightarrow \alpha = 1$

SECTION-4 : (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered;

Zero Marks : 0 In all other cases.

17. For $x \in \mathbb{R}$, then number of real roots of the equation $3x^2 - 4|x^2 - 1| + x - 1 = 0$ is ____.

Ans. (4)

Sol. $3x^2 + x - 1 = 4|x^2 - 1|$

If $x \in [-1, 1]$,

$3x^2 + x - 1 = -4x^2 + 4 \Rightarrow 7x^2 + x - 5 = 0$

say $f(x) = 7x^2 + x - 5$

$f(1) = 3; f(-1) = 1; f(0) = -1$

[Two Roots]

If $x \in (-\infty, -1] \cup [1, \infty)$

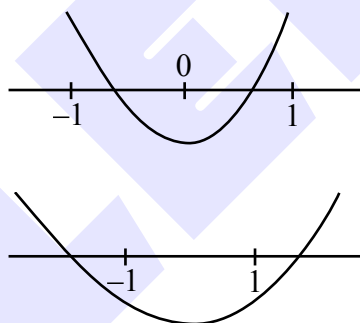
$3x^2 + x - 1 = 4x^2 - 4 \Rightarrow x^2 - x - 3 = 0$

Say $g(x) = x^2 - x - 3$

$g(-1) = -1; g(1) = -3$

[Two Roots]

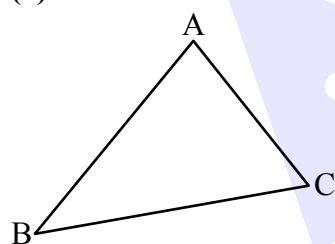
So total 4 roots.



18. In a triangle ABC, let $AB = \sqrt{23}$, $BC = 3$ and $CA = 4$. Then the value of $\frac{\cot A + \cot C}{\cot B}$ is ____.

Ans. (2)

Sol.



Given $c = \sqrt{23}$; $a = 3$; $b = 4$

$\cot A = \frac{\cos A}{\sin A} = \frac{b^2 + c^2 - a^2}{2bc \sin A}$

$= \frac{b^2 + c^2 - a^2}{2.2\Delta} \left\{ \Delta = \frac{1}{2} bc \sin A \right\}$

$\cot A = \frac{b^2 + c^2 - a^2}{4\Delta}$

Similarly, $\cot B = \frac{a^2 + c^2 - b^2}{4\Delta}$ & $\cot C = \frac{a^2 + b^2 - c^2}{4\Delta}$

$$\therefore \frac{\cot A + \cot C}{\cot B} = \frac{b^2 + c^2 - a^2 + a^2 + b^2 - c^2}{a^2 + c^2 - b^2} = \frac{2b^2}{a^2 + c^2 - b^2} = \frac{32}{16} = 2$$

19. Let \vec{u}, \vec{v} and \vec{w} be vectors in three-dimensional space, where \vec{u} and \vec{v} are unit vectors which are not perpendicular to each other and $\vec{u} \cdot \vec{w} = 1$, $\vec{v} \cdot \vec{w} = 1$, $\vec{w} \cdot \vec{w} = 4$

If the volume of the parallelepiped, whose adjacent sides are represented by the vectors \vec{u}, \vec{v} and \vec{w} , is $\sqrt{2}$, then the value of $|3\vec{u} + 5\vec{v}|$ is ____.

Ans. (7)

Sol. Given, $|\vec{u}| = 1$; $|\vec{v}| = 1$; $\vec{u} \cdot \vec{v} \neq 0$; $\vec{u} \cdot \vec{w} = 1$; $\vec{v} \cdot \vec{w} = 1$;

$$\vec{w} \cdot \vec{w} = |\vec{w}|^2 = 4 \Rightarrow |\vec{w}| = 2; [\vec{u} \ \vec{v} \ \vec{w}] = \sqrt{2}$$

$$\text{and } [\vec{u} \ \vec{v} \ \vec{w}]^2 = \begin{vmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} & \vec{u} \cdot \vec{w} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} & \vec{v} \cdot \vec{w} \\ \vec{w} \cdot \vec{u} & \vec{w} \cdot \vec{v} & \vec{w} \cdot \vec{w} \end{vmatrix} = 2$$

$$\Rightarrow \begin{vmatrix} 1 & \vec{u} \cdot \vec{v} & 1 \\ \vec{u} \cdot \vec{v} & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2$$

$$\Rightarrow \vec{u} \cdot \vec{v} = \frac{1}{2}$$

$$\text{So, } |3\vec{u} + 5\vec{v}| = \sqrt{9|\vec{u}|^2 + 25|\vec{v}|^2 + 2 \cdot 3 \cdot 5 \vec{u} \cdot \vec{v}}$$

$$= \sqrt{9 + 25 + 30 \left(\frac{1}{2} \right)} = \sqrt{49} = 7$$

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