PRACTICE PAPER-2 (SOLUTION)

CLASS – XII

SUBJECT : MATHEMATICS

SECTION – A

1. **(d)**

Given $A^2 = 3A$, $|A| \neq 0$, order of A is 3

$\therefore \quad |A^2| = |3A|$

$|A|^2 = 3^3|A|$ $\quad\quad (|A^2| = |A|^2 \;\; \& \;\; |KA| = K^n|A|)$

$|A| = 27$

2. **(b)**

As per question, direction cosines of the line :

$\ell = \cos 45° = \frac{1}{\sqrt{2}}$, $m = \cos 120° = \frac{-1}{2}$, $n = \cos\theta$

where θ is the angle, which line makes with positive z-axis.

We know that, $\ell^2 + m^2 + n^2 = 1$

$\Rightarrow \quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$

$\cos^2\theta = \frac{1}{4}$

$\Rightarrow \quad \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$ $\quad\quad$ (θ being acute)

$\Rightarrow \quad \theta = \frac{\pi}{3}$

3. **(b)**

$\vec{a} = \hat{i} + \alpha\hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} - \alpha\hat{j} + \hat{k}$

area of parallelogram = $|\vec{a} \times \vec{b}| = 8\sqrt{3}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \alpha & 3 \\ 3 & -\alpha & 1 \end{vmatrix} = \hat{i}(4\alpha) - \hat{j}(-8) + \hat{k}(-4\alpha)$$

$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{64 + 32\alpha^2} = 8\sqrt{3}$

$\Rightarrow 2 + \alpha^2 = 6 \Rightarrow \alpha^2 = 4$

$\therefore \quad \vec{a} \cdot \vec{b} = 3 - \alpha^2 + 3 = 2$

4. **(c)**

$$M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$$

$$|M| = 6 \text{ , adj } M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$$

Now $M^{-1} = \frac{\text{adj} M}{|M|}$

$$\therefore \; M^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$$

5. **(d)**

Let $I = \int_0^{\pi} \cos x \, e^{\sin x} \, dx \quad \ldots(i)$

$I = \int_0^{\pi} -\cos x \, e^{\sin x} dx \quad \ldots(ii) \quad$ (By property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$)

Adding (i) and (ii)

2I = 0

I = 0

6. **(b)**

$y = \sin(m \sin^{-1} x) \quad \ldots(i)$

$\Rightarrow \frac{dy}{dx} = \cos(m \sin^{-1} x) \times \left(\frac{m}{\sqrt{1-x^2}}\right)$ (Differentiating w.r.t. x)

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = m\cos(m\sin^{-1} x)$

(Again differentiating w.r.t. x)

$$\therefore \quad \sqrt{1-x^2}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right).\frac{1}{2\sqrt{1-x^2}}.(-2x) = -m^2 \sin(m\sin^{-1} x) \times \frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0 \quad$ (from (i))

7. **(d)**

Given differential equation is $\left(\frac{d^2y}{dx^2}\right)^5 + 4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\left(\frac{d^3y}{dx^3}\right)} + \frac{d^3y}{dx^3} = x^2 - 1$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^5\left(\frac{d^3y}{dx^3}\right) + 4\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{d^3y}{dx^3}\right)^2 = (x^2-1)\left(\frac{d^3y}{dx^3}\right)$$

The highest order derivative in the given equation is $\frac{d^3y}{dx^3}$ and its highest power is 2 Therefore

m = 3 and n = 2.

m + n = 3 + 2 = 5

8. (d)

Given that $\int \frac{f(x)dx}{\log\sin x} = \log\left[\log(\sin x)\right]$

Differentiating both sides w.r.t. x, we get

$\frac{f(x)}{\log\sin x} = \frac{\cot x}{\log\sin x} \Rightarrow f(x) = \cot x.$

9. (a)

Given, $\begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0$

$\Rightarrow x(x^2 - 2) - 2(x - 3) + 3(2 - 3x) = 0$

$\Rightarrow x^3 - 2x - 2x + 6 + 6 - 9x = 0$

$\Rightarrow x^3 - 13x + 12 = 0$

$\therefore (x + 4)(x^2 - 4x + 3) = 0$ $\quad [\because x = -4 \text{ is a root}]$

$\Rightarrow (x + 4)(x - 1)(x - 3) = 0$

Hence ; the sum of other two roots = 1 + 3 = 4

10. (c)

The corner points of the feasible region are (0, 0), (3, 0), (3, 2), (2, 3) and (0, 3)

11. (c)

Given, x dy – ydx = 0

$\Rightarrow xdy = ydx \Rightarrow \frac{1}{y}dy = \frac{1}{x}dx$

Integrating both sides

$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x}$

logy = logx + logc

y = cx $\quad \therefore$ it represents a straight line passing through origin.

12. (c)

Given ; E and F are independent i.e. $P(E \cap F) = P(E).P(F)$

$P(E \cup F) = P(E) + P(F) - P(E).P(F)$

$\Rightarrow \quad 0.5 = 0.3 + P(F) - 0.3 \times P(F)$

$\Rightarrow \quad 0.7\, P(F) = 0.2$

$\therefore \quad P(F) = \frac{2}{7}$

Now ; P(E/F) – P(F/E)

$= \frac{P(E \cap F)}{P(F)} - \frac{P(F \cap E)}{P(E)} = \frac{P(E).P(F)}{P(F)} - \frac{P(E).P(F)}{P(E)} = P(E) - P(F) = \frac{3}{10} - \frac{2}{7} = \frac{1}{70}$

13. **(c)**

$\because$ B = A' $\Rightarrow$ |B| = |A'| = |A|

$\therefore$ |AB| = |A||B| = (– 4)(– 4) = 16

14. **(b)**

Given, $|\vec{a}| = |\vec{b}|$, $|\vec{a}\times\vec{b}| = |\vec{a}|$ and $\vec{a}\perp\vec{b}$

Now, $|\vec{a}\times\vec{b}| = |\vec{a}| \Rightarrow |\vec{a}||\vec{b}|\sin 90° = |\vec{a}| \Rightarrow |\vec{b}| = 1 = |\vec{a}|$

$\therefore$ $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors.

Let θ be the angle between $(\vec{a}+\vec{b}+(\vec{a}\times\vec{b}))$ and $\vec{a}$

Let $\vec{a} = \hat{i}$, $\vec{b} = \hat{j}$

$\therefore$ $\vec{a}\times\vec{b} = \hat{k}$

$$\cos\theta = \frac{(\hat{i}+\hat{j}+\hat{k}).\hat{i}}{\sqrt{3}\ \sqrt{1}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

15. **(b)**

Given ; Z = px + qy

Since the minimum value of Z occurs at (3, 0) and (1, 1) :

$\Rightarrow$ 3p = p + q or p = $\frac{q}{2}$

16. **(c)**

Given, $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

$$KI - 8P = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} - \begin{bmatrix} 16 & -8 \\ 40 & -24 \end{bmatrix} = \begin{bmatrix} K-16 & 8 \\ -40 & K+24 \end{bmatrix} \quad ....(i) \quad \left(\because I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$P^2 = P\times P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$$

$$P^3 = P^2\times P = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix}$$

$$P^6 = P^3\times P^3 = \begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix} \times \begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix} \quad ...(ii)$$

from (i) & (ii)

$\because$ P^6 = KI – 8P

$$\begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix} = \begin{bmatrix} K-16 & 8 \\ -40 & K+24 \end{bmatrix} \Rightarrow -11 = K - 16 \Rightarrow K = 5$$

17. (a)

Given, $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}, \quad \overrightarrow{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$

Area of $\Delta OAB = \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}|$

Now, $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = 8\hat{i} - 10\hat{j} + 4\hat{k}$

$\Rightarrow |\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{64 + 100 + 16} = \sqrt{180}$

So area of $\Delta OAB = \frac{1}{2}\sqrt{180} = 3\sqrt{5}$ sq. units

18. (b)

Given that $f(x) = \begin{cases} \frac{|x-1|}{1-x} + a, & x > 1 \\ a + b, & x = 1 \\ \frac{|x-1|}{1-x} + b, & x < 1 \end{cases}$

$\because$ f(x) is continuous at x = 1; therefore, $\lim_{h \to 0} f(1+h) = \lim_{h \to 0} f(1-h) = f(1)$

$\because$ f(1) = a + b (given)

RHL = $\lim_{h \to 0} f(1 + h) = \lim_{h \to 0} \frac{|1+h-1|}{1-(1+h)} + a = -1 + a$

$\Rightarrow a + b = -1 + a \Rightarrow b = -1$

LHL = $\lim_{h \to 0} f(1 - h) = \lim_{h \to 0} \frac{|1-h-1|}{1-(1-h)} + b = 1 + b$

$\therefore$ a + b = 1 + b $\Rightarrow$ a = 1

19. (a)

Given equations $3\ell + m + 5n = 0$ ...(i)

$6mn - 2n\ell + 5\ell m = 0$

$m(6n + 5\ell) - 2n\ell = 0$ ...(ii)

From (i) $m = -3\ell - 5n$ ...(iii)

Putting in (ii) $\Rightarrow -(6n + 5\ell)(3\ell + 5n) - 2n\ell = 0$

$\Rightarrow 30n^2 + 45\ell n + 15\ell^2 = 0$

$\Rightarrow 2n^2 + 3n\ell + \ell^2 = 0$

$\Rightarrow \quad (2n + \ell)(n + \ell) = 0$

$\therefore$ either $\ell = -2n$ or $\ell = -n$

$m = n$ or $m = -2n$ from (iii)

$\therefore$ Direction numbers of the two lines are

$< -2n, n, n >$ and $< -n, -2n, n >$

i.e., $< -2, 1, 1 >$ and $< -1, -2, 1 >$

if θ is the acute angle between the lines, then

$$\cos\theta = \frac{|(-2)\times(-1) + 1\times(-2) + 1\times 1|}{\sqrt{4+1+1}\sqrt{1+4+1}} = \frac{1}{6} \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

Hence, Both A and R are true and R is the correct explanation of A

20. **(d)**

Assertion : Domain of $\sec^{-1}x$ is $(-\infty, -1] \cup [1, \infty)$

$\therefore \sec^{-1}(2x + 1)$ is meaningful, if

$2x + 1 \geq 1$ or $2x + 1 \leq -1$

$x \geq 0$ or $x \leq -1$

$\therefore x \in (-\infty, -1] \cup [0, \infty)$

So, Assertion is false

$$\sec^{-1}(-2) = \pi - \sec^{-1}(2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

So, reason is true.

SECTION – B

21. Let angle between $\vec{b}$ and $\vec{c}$ is α and given $|\vec{b}\times\vec{c}| = \sqrt{15}$

$\Rightarrow |\vec{b}|\ |\vec{c}| \sin\alpha = \sqrt{15}$

$\sin\alpha = \frac{\sqrt{15}}{4}$; $\therefore \cos\alpha = \frac{1}{4}$ **[½]**

Now, $\vec{b} - 2\vec{c} = \lambda\vec{a}$

$\Rightarrow |\vec{b} - 2\vec{c}|^2 = \lambda^2 |\vec{a}|^2$ $\left[\because \vec{a}^2 = |\vec{a}|^2\right]$

$|\vec{b}|^2 + 4|\vec{c}|^2 - 4\left(\vec{b}.\vec{c}\right) = \lambda^2 |\vec{a}|^2$ **[1]**

$16 + 4 - 4\{|\vec{b}||\vec{c}|\cos\alpha\} = \lambda^2$

$16 + 4 - 4\times 4\times 1\times\frac{1}{4} = \lambda^2 \Rightarrow \lambda^2 = 16 \Rightarrow \lambda = \pm 4.$ **[½]**

OR

Here, $\ell = \cos\theta, m = \cos\beta, n = \cos\theta$, $(\because \ell = n)$

Now, $\ell^2 + m^2 + n^2 = 1 \Rightarrow 2\cos^2\theta + \cos^2\beta = 1$ **[½]**

$\Rightarrow\ 2\cos^2\theta = \sin^2\beta$ [Given, $\sin^2\beta = 3\sin^2\theta$]

$\Rightarrow 2\cos^2\theta = 3\sin^2\theta$

$\Rightarrow 2\cos^2\theta - 3(1 - \cos^2\theta) = 0$

$\Rightarrow 5\cos^2\theta = 3,$

$\therefore \cos^2\theta = \dfrac{3}{5}$ **[1½]**

22. Given, y = x sin y; differentiating both sides w.r.t. x.

$$\frac{dy}{dx} = x\cos y\frac{dy}{dx} + \sin y.1 \Rightarrow \frac{dy}{dx} - x\cos y\frac{dy}{dx} = \sin y$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\sin y}{1 - x\cos y}$ **[1]**

or $x\dfrac{dy}{dx} = \dfrac{x\sin y}{1 - x\cos y}$, (multiplying both sides by x)

$\Rightarrow x\dfrac{dy}{dx} = \dfrac{y}{1 - x\cos y}$ $\quad(\because x\sin y = y)$ **[1]**

23. Let V, S and r denote the volume, surface area and radius of the salt ball respectively at any instant t.

Then $V = \dfrac{4}{3}\pi r^3$ and $S = 4\pi r^2$

It is given that the rate of decrease of the volume V is proportional to the surface area S.

i.e. $\dfrac{dV}{dt} \propto S$ or $\dfrac{dV}{dt} = -KS$, where K > 0 is the constant of proportionality.

$\dfrac{dV}{dt} = -KS$ **[1]**

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -K(4\pi r^2)$$

$$\Rightarrow \quad 4\pi r^2\frac{dr}{dt} = -4\pi K r^2$$

$$\Rightarrow \quad \frac{dr}{dt} = -K$$

So, r decrease with a constant rate **[1]**

24. Let required vector $\vec{c} = \lambda(\vec{a} \times \vec{b})$

$$\Rightarrow \quad \vec{c} = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 2 \end{vmatrix}$$

or $\quad \vec{c} = \lambda(\hat{i} - 11\hat{j} - 7\hat{k}) \quad \ldots(i)$ **[1]**

$\Rightarrow \quad |\vec{c}| = |\lambda|\sqrt{1+121+49}$

$\Rightarrow \quad \sqrt{171} = |\lambda|\sqrt{171} \quad$ (given $|\vec{c}| = \sqrt{171}$)

$\Rightarrow \quad \lambda = \pm 1$ put in equation (i), we have

$\vec{c} = \pm(\hat{i} - 11\hat{j} - 7\hat{k})$ **[1]**

25. $\sin^{-1}\left[\cos\left\{\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\}\right]$

$= \sin^{-1}\left\{\cos\left(\frac{-\pi}{3}\right)\right\}$ $\quad [\because \sin^{-1}(-x) = -\sin^{-1}x]$ **[½]**

$= \sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$ $\quad [\because \cos(-\theta) = \cos\theta]$ **[½]**

$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ **[1]**

OR

Given; $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$

$$\therefore f(x) = \begin{cases} \frac{x}{1+x} & ; \quad x \geq 0 \\ \frac{x}{1-x} & ; \quad x < 0 \end{cases}$$

For one-one function: Let $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$$

Case I : When x_1 & x_2 are positive:

$f(x_1) = f(x_2)$

$\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \Rightarrow x_1 = x_2$ **[½]**

Case II : When x_1 & x_2 are negative:

$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1 = x_2$ **[½]**

Case III : When $x_1 > 0$ and $x_2 < 0$

We have $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1-x_2}$

$\Rightarrow x_1 - x_1 x_2 = x_2 + x_1 x_2$

$\Rightarrow x_1 - x_2 = 2x_1 x_2$

This is not possible when $x_1 > 0$ and $x_2 < 0$

$\therefore x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$\therefore$ f is one-one

SECTION – C

26. Let $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$ **[½]**

When $x = \frac{\pi}{6}$, $t = \frac{(1-\sqrt{3})}{2}$ & when $x = \frac{\pi}{3}$, $t = \frac{(\sqrt{3}-1)}{2}$

also, $\sin x - \cos x = t \Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$ **[½]**

$\Rightarrow \sin 2x = 1 - t^2$

$\therefore I = \int_{\frac{(1-\sqrt{3})}{2}}^{\frac{(\sqrt{3}-1)}{2}} \frac{1}{\sqrt{1-t^2}} dt$

$= \left[\sin^{-1} t\right]_{(1-\sqrt{3})/2}^{(\sqrt{3}-1)/2}$ **[1]**

$= \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$

$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ **[1]**

27. Let E_1 and E_2 be the events of getting a total 10 by A and B respectively

favorable outcomes of total 10 is {(6, 4),(5, 5),(4, 6)}

Total outcomes = $6^2 = 36$

Now $P(E_1) = P(E_2) = \frac{3}{36} = \frac{1}{12}$

$P(\bar{E}_1) = P(\bar{E}_2) = \frac{11}{12}$ **[1]**

P(A wins) $= P(E_1) + P(\bar{E}_1)P(\bar{E}_2)P(E_1) + ...$

$$= \frac{1}{12} + \frac{1}{12} \times \left(\frac{11}{12}\right)^2 + \frac{1}{12} \times \left(\frac{11}{12}\right)^4 + ...$$

$$= \frac{1}{12}\left[\underbrace{1 + \left(\frac{11}{12}\right)^2 + \left(\frac{11}{12}\right)^4 + ...}_{\text{Infinite G.P.}}\right] = \frac{1}{12}\left[\frac{1}{1 - \frac{11^2}{12^2}}\right] \qquad \left(\because S_\infty = \frac{a}{1-r}\right)$$ **[1]**

$$P(A \text{ wins}) = \frac{1}{12} \times \left[\frac{12^2}{144 - 121}\right] = \frac{12}{23}$$

P(B wins) $= 1 - P(\text{win } A) = 1 - \frac{12}{23} = \frac{11}{23}$ **[1]**

28. Let P be the principal at any time t. According to the given problem,

$$\frac{dP}{dt} = \left(\frac{5}{100}\right) \times P$$

or $\frac{dP}{dt} = \frac{P}{20}$ .....(i) **[1]**

Separating the variables in equation (i), we get

$$\frac{dP}{P} = \frac{dt}{20} \qquad .....(ii)$$

Integrating both sides of equation (ii), we get

$$\log P = \frac{t}{20} + C_1$$

or $P = e^{\frac{t}{20}} . e^{C_1}$

or $P = Ce^{\frac{t}{20}}$ (where $e^{C_1} = C$) ....(iii) **[1]**

Now P = 1000, when t = 0

Substituting the values of P and t in (iii), we get C = 1000. Therefore, equation (iii), gives

$$P = 1000\, e^{\frac{t}{20}}$$

Let t years be the time required to double the principal. Then

$2000 = 1000 e^{\frac{t}{20}} \Rightarrow t = 20 \log_e 2$ **[1]**

OR

The given differential equation can be written as $\frac{dy}{dx}=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)}$ ....(i)

Let $F(x, y)=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)}=\frac{y}{x}+\sec\left(\frac{y}{x}\right)=f\left(\frac{y}{x}\right)$ **[1]**

Thus, F(x, y) is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation.

To solve it we make the substitution

$y = vx$ ....(ii)

Differentiating equation (ii) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{....(iii)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{\cos v}$

$\Rightarrow \quad \cos v\, dv = \frac{1}{x}dx$ (Separating variables) **[1]**

Integrate both sides

$\Rightarrow \quad \int \cos v\, dv = \int \frac{1}{x}dx$

$\Rightarrow \quad \sin v = \log|x| + \log|C|$

$\Rightarrow \quad \sin v = \log|Cx|$

Replacing v by $\frac{y}{x}$, we get $\sin\left(\frac{y}{x}\right) = \log|Cx|$ **[1]**

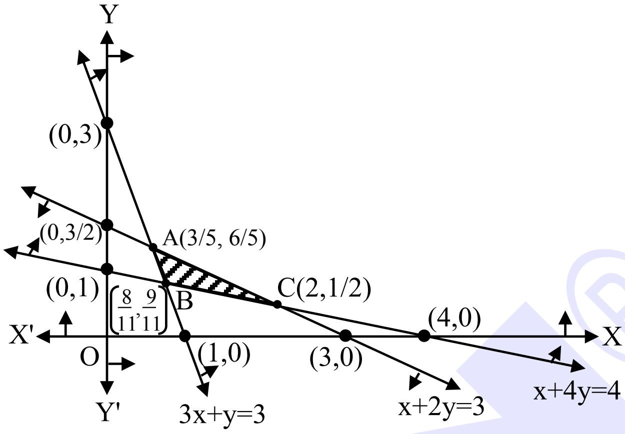
Which is the general solution of the differential equation (i).

29. Minimize Z = 6x + 21y

subject to constraints

$x + 2y \le 3,\ x + 4y \ge 4$

$3x + y \ge 3,\ x \ge 0,\ y \ge 0$



[1]

Corner points	**Z = 6x+ 21y**
$A\left(\frac{3}{5},\frac{6}{5}\right)$	$Z_A = \frac{144}{5} = 28.8$
$B\left(\frac{8}{11},\frac{9}{11}\right)$	$Z_B = \frac{237}{11} = 21.54$
$C\left(2,\frac{1}{2}\right)$	$Z_C = \frac{45}{2} = 22.5$

[1½]

Since, feasible region is bounded, so

Minimum value of Z is 21.54 at point $B\left(\frac{8}{11},\frac{9}{11}\right)$ **[½]**

30. $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}dx$

$= \int_0^{\pi/2} \frac{\cos^2(x/2)-\sin^2(x/2)}{2\cos^2(x/2)+2\sin(x/2)\cos(x/2)}dx$ **[½]**

$= \frac{1}{2}\int_0^{\pi/2} \frac{1-\tan^2(x/2)}{1+\tan(x/2)}dx \qquad \left[\text{Divide Nr \& Dr by } \cos^2\frac{x}{2}\right]$

$= \frac{1}{2}\int_0^{\pi/2}\left[1-\tan\left(\frac{x}{2}\right)\right]dx$ **[½]**

$= \frac{1}{2}\left[x+2\log\cos\frac{x}{2}\right]_0^{\pi/2}$ **[1½]**

$$= \frac{\pi}{4} + \log \frac{1}{\sqrt{2}}.$$ **[½]**

OR

Let $I = \int_0^{\pi/2} \left| \sin\left(x - \frac{\pi}{4}\right) \right| dx$

$x - \frac{\pi}{4}$ is *–ve* when $x \le \frac{\pi}{4}$ and *+ve* when $x > \frac{\pi}{4}$ **[½]**

$$\therefore I = -\int_0^{\pi/4} \sin\left(x - \frac{\pi}{4}\right) dx + \int_{\pi/4}^{\pi/2} \sin\left(x - \frac{\pi}{4}\right) dx$$ **[½]**

$$= \left[\cos\left(x - \frac{\pi}{4}\right)\right]_0^{\pi/4} - \left[\cos\left(x - \frac{\pi}{4}\right)\right]_{\pi/4}^{\pi/2} = \left(1 - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} - 1\right)$$ **[1½]**

$$= 2 - \sqrt{2}.$$ **[½]**

31. Let $I = \int \frac{3x+5}{(x^3 - x^2 - x + 1)} dx$

Let $\frac{3x+5}{(x^3 - x^2 - x + 1)} = \frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$ **[1]**

$\Rightarrow (3x + 5) = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$ …(1)

Put x = 1 in equation (1) we get B = 4

Put x = –1 in equation (1) we get C = $\frac{1}{2}$

Comparing the coefficient of 'x^2' on both sides of (1); we get:

$A + C = 0 \Rightarrow A = -C = -\frac{1}{2}$ **[1]**

$$\Rightarrow \frac{(3x+5)}{(x^3 - x^2 - x + 1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\therefore I = \int \frac{(3x+5)}{(x^3 - x^2 - x + 1)} dx$$

$$= \frac{-1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2} + \frac{1}{2}\int \frac{dx}{(x+1)}$$

$$= \frac{-1}{2}\log|x-1| - \frac{4}{(x-1)} + \frac{1}{2}\log|x+1| + C$$ **[1]**

SECTION – D

32. Given $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{bmatrix}$

$\Rightarrow \quad |A| = 3(12 - 6) - 4(0 + 3) + 2(0 - 2)$

$= 18 - 12 - 4 = 2 \neq 0$ **[1]**

Hence, A^{-1} exists.

Now, co-factor are given as:

$C_{11} = 6, C_{12} = -3, C_{13} = -2,$

$C_{21} = -28, C_{22} = 16, C_{23} = 10,$

$C_{31} = -16, C_{32} = 9, C_{33} = 6$ **[1]**

So, $\text{adj } A = \begin{bmatrix} 6 & -3 & -2 \\ -28 & 16 & 10 \\ -16 & 9 & 6 \end{bmatrix}^T$

$= \begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix}$ **[1]**

$\because A^{-1} = \frac{1}{|A|} \cdot \text{adj} A \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix}$ .....(i)

The given system of linear equation are :

$3x + 4y + 2z = 8$

$2y - 3z = 3$

$x - 2y + 6z = -2$

It can be represented as :

$\Rightarrow \begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$

$\Rightarrow AX = B$

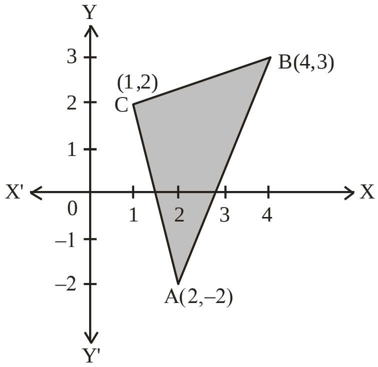
$\Rightarrow X = A^{-1} B$ **[1]**

$X = \frac{1}{2}\begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$ (From equation (i))

$= \frac{1}{2}\begin{bmatrix} 48 - 84 + 32 \\ -24 + 48 - 18 \\ -16 + 30 - 12 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 \\ 6 \\ 2 \end{bmatrix}$ $\Rightarrow x = -2, y = 3, z = 1$ **[1]**

33.



[Correct Fig. 1 Mark]

Line AB is : $y=\frac{5}{2}x-7;\ x=\frac{2}{5}(y+7)$, line BC is: $y=\frac{1}{3}(x+5)\Rightarrow x=3y-5$

Line AC is: $y = -4x + 6$; $x=\frac{y-6}{-4}$ **[1]**

Required area = $\left[\int_{-2}^{3}(\text{lineAB})dy\right]-\left[\int_{2}^{3}(\text{lineBC})dy+\int_{-2}^{2}(\text{lineAC})dy\right]$

$\Rightarrow\quad \left[\frac{2}{5}\int_{-2}^{3}(y+7)dy\right]-\left[\int_{2}^{3}(3y-5)dy-\frac{1}{4}\int_{-2}^{2}(y-6)dy\right]$ **[1]**

$=\frac{2}{5}\left[\left(\frac{y^2}{2}+7y\right)_{-2}^{3}\right]-\left[\left(\frac{3y^2}{2}-5y\right)_{2}^{3}-\frac{1}{4}\left(\frac{y^2}{2}-6y\right)_{-2}^{2}\right]$

$=\frac{2}{5}\left[\left(\frac{9}{2}+21\right)-(2-14)\right]-\left[\left\{\left(\frac{27}{2}-15\right)-(6-10)\right\}-\frac{1}{4}\{(2-12)-(2+12)\}\right]$

$=\frac{2}{5}\left[\frac{9}{2}+33\right]-\left[\left(\frac{27}{2}-11\right)-\frac{1}{4}(-24)\right]$

$=\left(\frac{2}{5}\times\frac{75}{2}\right)-\left(\frac{5}{2}+6\right)=15-\frac{17}{2}=\frac{13}{2}$ square units **[2]**

34. Given lines are

$\vec{r}=3\hat{i}+8\hat{j}+3\hat{k}+\lambda(3\hat{i}-\hat{j}+\hat{k})$ ….(i)

and $\vec{r}=-3\hat{i}-7\hat{j}+6\hat{k}+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$ ….(ii)

Equation of lines (i) and (ii) respectively in cartesian form are

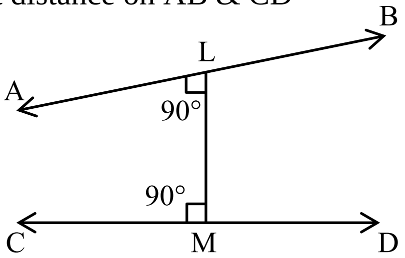
Let AB : $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=\lambda$ ….(iii)

and CD : $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=\mu$ ….(iv) **[1]**

$\therefore$ Also let L & M be end points of line of shortest distance on AB & CD

$\therefore$ Co-ordinates of L is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$ **[1]**

and $M(-3\mu - 3, 2\mu - 7, 4\mu + 6)$



Direction ratio of LM are

$3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$ **[1]**

Since LM $\perp$ AB

$\therefore \quad 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$

$\Rightarrow \quad 11\lambda + 7\mu = 0$ ....(v)

Also LM $\perp$ CD

$\therefore \quad -3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

$\Rightarrow \quad -7\lambda - 29\mu = 0$ ....(vi)

Solving (v) and (vi) we get

$\lambda = 0$ and $\mu = 0$

$\therefore \quad L(3, 8, 3)$ and $M(-3, -7, 6)$ **[1]**

Hence shortest distance LM = $\sqrt{(3+3)^2+(8+7)^2+(3-6)^2} = \sqrt{36+225+9}$

$= 3\sqrt{30}$ units **[½]**

Vector equation of LM is

$\vec{r} = 3\hat{i}+8\hat{j}+3\hat{k}+t(6\hat{i}+15\hat{j}-3\hat{k})$ **[½]**

OR

At t = 6

Position of first insect is at $A(6 + 6, 8 - 6, 3 + 6) \Rightarrow A(12, 2, 9)$ **[½]**

Position of 2nd insect at $B(1 + 6, 2 + 6, 2(6)) \Rightarrow B(7, 8, 12)$ **[½]**

$\therefore$ Distance between insects after 6 min. = $\sqrt{(12-7)^2+(2-8)^2+(9-12)^2}$

$= \sqrt{25+36+9} = \sqrt{70}$ inches **[½]**

For the closest distance between two insects

Given equation of lines are $\ell_1 : x = 6 + t, y = 8 - t, z = 3 + t$

$\Rightarrow \vec{r} = (6\hat{i}+8\hat{j}+3\hat{k})+t(\hat{i}-\hat{j}+\hat{k})$ ...(i) **[½]**

and $\ell_2 : x = 1 + t, y = 2 + t, z = 2t$

$\Rightarrow \quad \vec{r} = (\hat{i}+\hat{j})+t(\hat{i}+\hat{j}+2\hat{k})$ **[½]**

On Comparing it with $\vec{r} = \vec{a}+\lambda\vec{b}$

$\vec{a}_1 = 6\hat{i}+8\hat{j}+3\hat{k},\ \vec{b}_1 = \hat{i}-\hat{j}+\hat{k}$

$\vec{a}_2 = \hat{i}+\hat{j},\ \vec{b}_2 = \hat{i}+\hat{j}+2\hat{k}$

Now $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -3\hat{i} - \hat{j} + 2\hat{k}$ [½]

$\vec{a}_2 - \vec{a}_1 = -5\hat{i} - 7\hat{j} - 3\hat{k}$ [½]

Shortest distance = $\left|\dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\dfrac{(-5\hat{i} - 7\hat{j} - 3\hat{k}).(-3\hat{i} - \hat{j} + 2\hat{k})}{|-3\hat{i} - \hat{j} + 2\hat{k}|}\right|$ [1]

$= \left|\dfrac{15 + 7 - 6}{\sqrt{9 + 1 + 4}}\right| = \dfrac{16}{\sqrt{14}} = \dfrac{8}{7}\sqrt{14}$ inches [½]

35. **Reflexive :**

Let (a, b) be an arbitrary element of N × N, $\forall$ a, b $\in$ N. Then

$\Rightarrow$ (a , b) R (a, b)

$\Rightarrow$ ab(b + a) = ba(a + b) [by commutativity of addition and multiplication on N]

$\Rightarrow$ L.H.S. = R.H.S.

Thus, (a, b) R (a, b) for all (a, b) $\in$ N × N. So R is reflexive on N × N. [1]

Symmetric :

Let (a, b), (c, d) $\in$ N × N be such that (a, b) R (c, d). Then,

$\Rightarrow$ ad(b + c) = bc(a + d)

$\Rightarrow$ cb(d + a) = da(c + b)

$\Rightarrow$ (c, d) R (a, b) [1½]

Thus, (a, b) R (c, d) $\Rightarrow$ (c, d) R (a, b) for all (a, b), (c, d) $\in$ N × N

So, R is symmetric on N × N.

Transitive :

Let (a, b), (c, d), (e, f) $\in$ N × N such that (a, b) R (c, d) and (c, d) R (e, f). Then,

(a, b) R (c, d) $\Rightarrow$ ad(b + c) = bc(a + d) $\Rightarrow \dfrac{b+c}{bc} = \dfrac{a+d}{ad} \Rightarrow \dfrac{1}{c} + \dfrac{1}{b} = \dfrac{1}{d} + \dfrac{1}{a}$ ...(i)

and, (c, d) R (e, f) $\Rightarrow$ cf(d + e) = de(c + f) $\Rightarrow \dfrac{d+e}{de} = \dfrac{c+f}{cf} \Rightarrow \dfrac{1}{e} + \dfrac{1}{d} = \dfrac{1}{f} + \dfrac{1}{c}$ ...(ii)

Adding (i) and (ii), we get

$$\left(\frac{1}{c} + \frac{1}{b}\right) + \left(\frac{1}{e} + \frac{1}{d}\right) = \left(\frac{1}{d} + \frac{1}{a}\right) + \left(\frac{1}{f} + \frac{1}{c}\right)$$

$\Rightarrow \dfrac{1}{b} + \dfrac{1}{e} = \dfrac{1}{a} + \dfrac{1}{f} \Rightarrow \dfrac{b+e}{be} = \dfrac{a+f}{af}$

$\Rightarrow$ af(b + e) = be (a + f) $\Rightarrow$ (a, b) R (e, f) [2]

So, R is transitive on N × N.

Hence; R being reflexive, symmetric and transitive; is an equivalence relation on N × N. [½]

OR

Given, $A = \{x \in Z : 0 \le x \le 12\}$ and $R = \{(a, b) : a,b \in A, |a - b| \text{ is divisible by } 4\}$

(i) **For Reflexive relation :**

Let $a \in A$

Now, $|a - a| = 0$, which is divisible by 4

So, $(a, a) \in R \ \forall \ a \in A$

Hence, R is reflexive. **[1]**

(ii) **For Symmetric relation:**

Let $a, b \in A$ such that $(a, b) \in R$

i.e. $|a - b|$ is divisible by 4.

$\Rightarrow$ $|-(b - a)| = |b - a|$ is also divisible by 4.

So, $(b, a) \in R$.

Hence, R is symmetric. **[1]**

(iii) **For Transitive relation:**

Let $a, b, c \in A$ such that (a, b) and $(b, c) \in R$

i.e. $|a - b|$ & $|b - c|$ is divisible by 4.

Let $|a - b| = 4k_1$

& $|b - c| = 4k_2$

$\Rightarrow$ $(a - b) = \pm 4k_1$ ....(i)

& $(b - c) = \pm 4k_2$ ....(ii)

Adding equations (i) & (ii);

$\Rightarrow$ $(a - b) + (b - c) = \pm 4k_1 \pm 4k_2 = \pm 4(k_1 + k_2)$

$\Rightarrow$ $a - c$ is divisible by 4.

$\Rightarrow$ $|a - c|$ is divisible by 4.

So, $(a, c) \in R$

Hence, R is transitive. **[2]**

R is an equivalence relation.

Further, let $(x, 1) \in R \ \forall \ x \in A$

$\Rightarrow$ $|x - 1|$ is divisible by 4

$\Rightarrow$ $x - 1 = 0, 4, 8$

$\Rightarrow$ $x = 1, 5, 9$

$\therefore$ Equivalence class of 1 = [1] = {1, 5, 9} **[½]**

[$\because$ The set of all elements related to 1 represents its equivalence class]

Now, we will find equivalence class of 2 i.e. [2]

Let $(x, 2) \in R \ \forall \ x \in A$

$\Rightarrow$ $|x - 2| = 0, 4, 8$

$\Rightarrow$ $x = 2, 6, 10$

Equivalence class of [2] = {2, 6, 10}. **[½]**

SECTION – E

36. (i) Given volume of cylinder $V = \frac{539}{2}$ cubic units

$\because V = \pi r^2 h = \frac{539}{2}$

$h = \frac{539}{2\pi r^2}$

Total surface area of the tank

$S = 2\pi rh + 2\pi r^2$

$S = 2\pi r\left(\frac{539}{2\pi r^2}\right) + 2\pi r^2 = \frac{539}{r} + 2\pi r^2$ square units **[1]**

(ii) $\because S = \frac{539}{r} + 2\pi r^2$

$\frac{dS}{dr} = -\frac{539}{r^2} + 4\pi r$ ....(i)

$= -\frac{539}{r^2} + \frac{4 \times 22r}{7}$

$= 11\left(\frac{-343 + 8r^3}{7r^2}\right)$

For critical points

$\frac{dS}{dr} = 11\left(\frac{-343 + 8R^3}{7r^2}\right) = 0$

$8r^3 = 343$

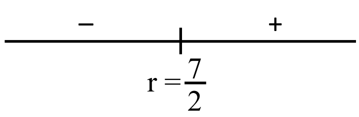
$r^3 = \frac{343}{8}$

$r = \frac{7}{2}$ unit **[1]**

(iii) By first derivative test

When $r < \frac{7}{2}$; $\frac{dS}{dr} < 0$

When $r > \frac{7}{2}$; $\frac{dS}{dr} > 0$



$\because \frac{dS}{dr}$ changes its sign from negative to positive at neighborhood of $r = \frac{7}{2}$

So, $r = \frac{7}{2}$ is point of minima **[1]**

$\therefore$ Surface area is minimum at $r = \frac{7}{2}$ and corresponding height

$h = \frac{539}{2\pi r^2} = \frac{539 \times 7 \times 2 \times 2}{2 \times 22 \times 7 \times 7} = 7$ unit **[1]**

OR

(iii) Again differentiate equation (i) w.r.t. 'r'

$$\frac{d^2S}{dr^2}=\frac{2\times 539}{r^3}+4\pi$$

$$\left.\frac{d^2S}{dr^2}\right|_{r=\frac{7}{2}}>0$$ **[1]**

So, S is minimum at r = $\frac{7}{2}$

h = $\frac{539}{2\pi r^2}$ = $\frac{539\times 7\times 2\times 2}{2\times 22\times 7\times 7}$ = 7 unit **[1]**

37. (i) Since the perimeter of the floor = 200 m

i.e. $2\times\pi\left(\frac{y}{2}\right)+2x$ = 200

⇒ πy + 2x = 200 ...(i) **[1]**

(ii) ∵ A = x × y

⇒ A = $x\left(\frac{200-2x}{\pi}\right)$ [using (i)]

⇒ A = $\frac{2}{\pi}(100x-x^2)$ **[1]**

(iii) ∵ A = $\frac{2}{\pi}(100x-x^2)$

⇒ $\frac{dA}{dx}=\frac{2}{\pi}(100-2x)$

⇒ $\frac{d^2A}{dx^2}=\frac{2}{\pi}\times(-2)=-\frac{4}{\pi}$ **[1]**

For maximum value of A

$\frac{dA}{dx}=0$

⇒ $\frac{2}{\pi}\times(100-2x)=0$

⇒ x = 50

Now, at x = 50, $\frac{d^2A}{dx^2}=\frac{-4}{\pi}<0$

i.e., A is maximum at x = 50

So, Maximum value of A = $\frac{2}{\pi}\left[100\times 50-(50)^2\right]$

= $\frac{2}{\pi}\left[5000-2500\right]=\frac{2}{\pi}\times 2500$

= $\frac{5000}{\pi}m^2$ **[1]**

OR

(iii) Let B is the area of whole floor including the semi-circular ends, Then

$$B = 2 \times \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 + xy$$

$$B = \frac{\pi}{4}y^2 + xy$$

$$B = \frac{\pi}{4}\left(\frac{200-2x}{\pi}\right)^2 + x\left(\frac{200-2x}{\pi}\right)$$ [using (i)] **[1]**

$$\Rightarrow \quad B = \frac{1}{4\pi}(200-2x)^2 + \frac{x}{\pi}(200-2x)$$

$$\Rightarrow \quad B = \frac{(200-2x)}{\pi}\left[\frac{200-2x}{4} + x\right]$$

$$\Rightarrow \quad B = \frac{(200-2x)}{\pi}\frac{(200+2x)}{4} = \frac{40000-4x^2}{4\pi} \quad ....(ii)$$

$$\Rightarrow \quad \frac{dB}{dx} = \frac{1}{4\pi}(-8x) \text{ and } \frac{d^2B}{dx^2} = -\frac{8}{4\pi}$$

For maximum value of B

$$\frac{dB}{dx} = 0 \Rightarrow \frac{1}{4\pi}(-8x) = 0 \Rightarrow x = 0$$

at x = 0, $\frac{d^2B}{dx^2} = -\frac{8}{4\pi} < 0$

i.e., B is maximum at x = 0

So, maximum value of B = $\frac{40000-x^2}{4\pi} = \frac{40000}{4\pi} = \frac{10000}{\pi}m^2$ [From equation (ii)]**[1]**

38. Let X be the random variable which represents number of tails.

Here X can be 0, 1 or 2

(i) Probability distribution is

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

[2]

(ii) S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Let E : at least 2 heads

E = {HHH, HHT, HTH, THH} **[½]**

and F : at most 2 head

F = {HHT, HTH, HTT, THH, THT, TTH, TTT}

$\therefore$ $E \cap F$ = {HHT, HTH, THH} **[½]**

Clearly $P(E \cap F) = \frac{3}{8}$ and $P(F) = \frac{7}{8}$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{3/8}{7/8} = \frac{3}{7}$$ **[1]**