SOLUTION

REGIONAL MATHEMATICAL OLYMPIAD – 2025 (RMO)

DATE : 16-Nov-2025

1. (a) Let $n \geq 3$ be an integer. Find a configuration of n lines in the plane which has exactly.

(i) $n - 1$ distinct points of intersection

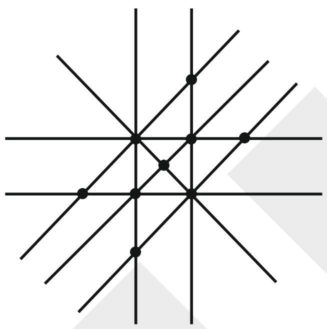
(ii) n distinct points of intersection.

(b) Give configuration of n lines that have exactly $n + 1$ distinct points of intersection for (i) $n = 8$ and (ii) $n = 9$.

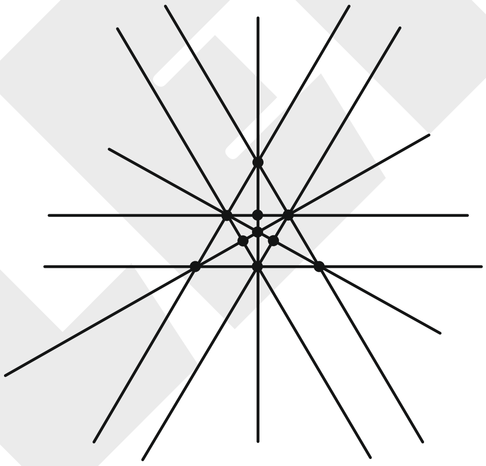
Sol. (a) (i) $n - 1$ parallel lines (non coincident) and 1 line not parallel to the rest.

(ii) $n - 1$ concurrent lines followed by 1 line not parallel to any other line not passing through the original point of concurrency.

(b)



n = 8 with 9 intersections



n = 9 with 10 intersection

(i) For $n = 8$

Consider 4 lines as sides of a square extended both sides. further consider the 2 diagonals extended as 2 more lines. Now, make 2 lines pass through vertices of the square on opposite ends of any 1 diagonal, parallel to the other diagonal.

(ii) For $n = 9$

Consider 3 lines as sides of a Δ extended both sides. Further consider the 3 medians extended as 3 more lines. Now, join the pair of mid points on each side of the original Δ, these will be parallel to the other side hence making 3 pairs of parallel lines.

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2. Let a, b, c be distinct nonzero real numbers satisfying

$$a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}.$$

Determine the value of $|a^2b+b^2c+c^2a|$.

Sol. $$a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}$$

(i) (ii) (iii)

from (i) and (ii)

$$a-b=\frac{2}{c}-\frac{2}{b}$$

$$(a-b)=2\left(\frac{b-c}{bc}\right) \quad \ldots(1)$$

Similarly

$$(b-c)=2\left(\frac{c-a}{ca}\right) \quad \ldots(2)$$

and $$(c-a)=2\left(\frac{a-b}{ab}\right) \quad \ldots(3)$$

From multiplication of eq (1), (2) and (3)

$$1=\frac{8}{a^2b^2c^2}$$

$$a^2b^2c^2=8 \qquad \Rightarrow \ abc=\pm\, 2\sqrt{2}$$

from eq(1) : $abc-b^2c=2(b-c)$ ....(4)

Similarly $abc-ac^2=2(c-a)$ ....(5)

$abc-ab^2=2(a-b)$ ....(6)

By adding eq (4), (5) and (6)

$$3abc=ab^2+bc^2+ca^2$$

$$ab^2+bc^2+ca^2=3\,(\pm 2\sqrt{2}\,)=\pm 6\sqrt{2}$$

$$|\,ab^2+bc^2+ca^2|=6\sqrt{2}$$

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Alternate

Let $a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}=k$

also let $f(x)=k-\frac{2}{x}$

Note : $f^3(a)=f\left(f\left(f(a)\right)\right)=f\left(f(c)\right)=f(b)=a$

$\Rightarrow\ f^3(a)=a$

$$f^2(x)=k-\frac{2}{k-\frac{2}{x}}=k-\frac{2x}{kx-2}=\frac{(k^2-2)x-2k}{(kx-2)}$$

$$f^3(x)=k-\frac{2(kx-2)}{(k^2-2)x-2k}=\frac{k(k^2-2)x-2k^2+4}{(k^2-2)x-2k}$$

if $f^3(x)=x$

$$\Rightarrow\ \frac{k(k^2-2)x-2k^2-2kx+4}{(k^2-2)x-2k}$$

if $(k^2-2)x-2k\neq 0$

$k(k^2-2)x-2k^2-2kx+4=x^2+(k^2-2)-2kx$

$\Rightarrow\ (k^2-2)(x^2-kx+2)=0$

This is an identify if $k^2=2$ or $k=\pm\sqrt{2}$

if $k^2\neq 2$ then $x^2-kx+2=0$

$\Rightarrow\ x=k-\frac{2}{x}$

$\Rightarrow\ x=f(x)$

This implies $a=b=c$ (Not possible)

Thus $f^3(x)=x$, $f(x)\neq x$, $(k^2-2)x-2k\neq 0\Rightarrow k^2=2$

if $(k^2-2)x-2k=0$

$\Rightarrow\ x=\frac{2k}{k^2-2},\ kx=2$

$\Rightarrow\ \frac{2k}{k^2-2}=\frac{2}{k}\Rightarrow k^2=k^2-2$ No solution.

if (a, b, c) exist such that $k=-\sqrt{2}$

then $(-a,-b,-c)$ give $k=\sqrt{2}$

hence w.l.o.g. let $k=\sqrt{2}$

$\Rightarrow\ ab = b\sqrt{2} - 2$

$bc = c\sqrt{2} - 2$

$ca = a\sqrt{2} - 2$

$\Rightarrow\ \Sigma ab = \sqrt{2}\,\Sigma a - 6$

Now $|a^2b + b^2c + c^2a|$

$= |(ab)a + (bc)b + (ca)c|$

$= \left|\left(\sqrt{2}b - 2\right)a + \left(\sqrt{2}c - 2\right)b + \left(\sqrt{2}a - 2\right)c\right|$

$= \left|\left(\sqrt{2}\Sigma ab\right) - 2\Sigma a\right|$

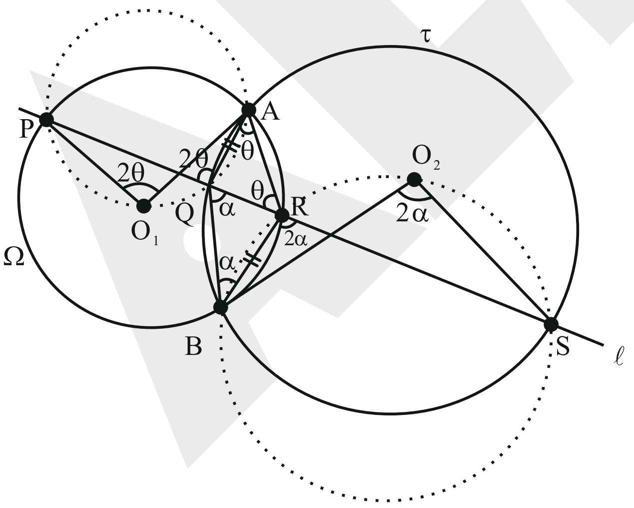
$= \left|\sqrt{2}\left(\sqrt{2}\Sigma a - 6\right) - 2\Sigma a\right|$

$= \left|2\Sigma a - 6\sqrt{2} - 2\sum a\right|$

$= 6\sqrt{2}$

3. Let Ω and Γ be circles centred at O_1, O_2 respectively. Suppose that they intersect in distinct points A, B. Suppose O_1 is outside Γ and O_2 is outside Ω. Let ℓ be a line not passing through A and B that intersects Ω at P, R and Γ at Q, S so that P, Q, R, S lie on the line in this order. Furthermore the points O_1, B lie on one side of ℓ and the points O_2, A lie on the other side of ℓ. Given that the points A, P, Q, O_1 are concyclic and B, R, S, O_2 are concyclic as well, prove that AQ = BR.

Sol.



To prove : AQ = BR

Let $\angle PRA = \theta$,

So, $\angle POA = 2\theta$ (from circle Ω)

$\angle POA = \angle PQA = 2\theta$ as POQA is cyclic

So, $\angle QAR = \theta$ as $\angle QAR = \angle PQA - \angle QRA$

Hence $QA = QR$ ...(1)

Let $\angle SQB = \alpha$

So, $\angle BO_2S = 2\alpha$ (from circle τ)

So $\angle BRS = \angle BO_2S = 2\alpha$ as BRO_2S is cyclic

$\angle QBR = \alpha$ as $\angle BRS = \angle BQS + \angle QBR$

Hence $QR = RB$ ...(2)

From equation (1) and (2)

$AQ = BR$. Proved.

4. Prove that there do not exist positive rational numbers x and y such that

$$x + y + \frac{1}{x} + \frac{1}{y} = 2025.$$

Sol. $x + y + \frac{1}{x} + \frac{1}{y} = 2025$

Let $x = \frac{a}{b}$, where $\gcd(a,b) = 1$

$y = \frac{c}{d}$, where $\gcd(c,d) = 1$

Now $x + \frac{1}{x} + y + \frac{1}{y} = 2025$

$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} = 2025$$

$$\frac{a^2+b^2}{ab} + \frac{c^2+d^2}{cd} = 2025$$

$$cd\,(a^2 + b^2) + ab\,(c^2 + d^2) = 2025\,abcd$$

Now since $\gcd(ab, a^2 + b^2) = \gcd(cd, c^2 + d^2) = 1$,

So, $ab \mid cd$ and $cd \mid ab$ implies $ab = cd$

So, $a^2 + b^2 + c^2 + d^2 = 2025\,ab$

Now, RHS $= 0 \pmod 3$, and any $n^2 = 0, 1 \pmod 3$,

So, for $a^2 + b^2 + c^2 + d^2$ to be $0 \pmod 3$, $\{a, b, c, d\} \in (0, 0, 0, 0) \pmod 3$ or $(1, 1, 1, 0) \pmod 3$, $(0, 0, 0, 0) \pmod 3$ not possible as $\gcd(a, b) = \gcd(c,d) = 1$ & $(1, 1, 1, 0) \pmod 3$ not possible as $ab = cd$.

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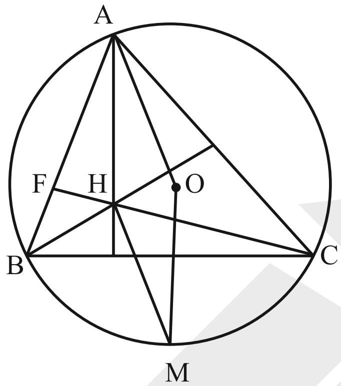
5. Let ABC be an acute angled triangle with AB < AC, orthocentre H and circumcircle Ω. Let M be the midpoint of minor are BC of Ω. Suppose that MH is equal to the radius of Ω. Prove that $\angle BAC = 60°$.

Sol.



AO = OM = MH = R (given)

Since OM $\perp$ BC (M is midpoint of minor arc BC)

and AH $\perp$ BC

$\Rightarrow$ OM || AH

Now OH < R as H lies inside acute ΔABC

$\Rightarrow$ $\angle$OMH < 60°

Since both O and H lie inside acute ΔABC

$\angle OAH < \angle BAC < 90°$

$\Rightarrow$ both opposite angles. $\angle$OAH and $\angle$OMH are acute.

$\Rightarrow$ AOM is a parallelogram

$\Rightarrow$ AH = OM = R

Let foot of $\perp$ from C to AB be F

$\Rightarrow$ AF = AC cos A

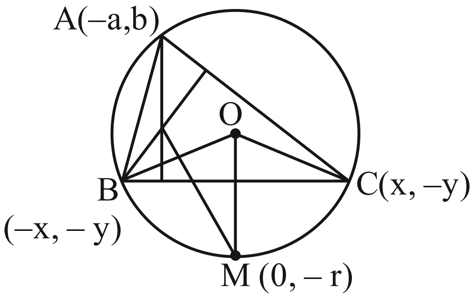
and $\dfrac{AF}{AH} = \sin B \Rightarrow AH = \dfrac{AC\cos A}{\sin A} = 2R\cos A$ (sine rule)

$\Rightarrow$ 2RcosA = R

$\Rightarrow$ $\cos A = \dfrac{1}{2}$

$\Rightarrow$ $\angle A = 60°$

Alternate:



Let O be the circumcenter be at the origin.

B = (–x, –y), c = (x, –y), A = (a, –b)

a, b, x, y, > 0 and $\frac{b}{a} > \frac{x}{y}$

this will ensure AB < AC, ΔABC is acute

$\Rightarrow$ circumradius $= r = \sqrt{a^2+b^2} = \sqrt{x^2+y^2}$

$\Rightarrow$ M = (0, – r)

Now H is orthocentre of ΔABC and centroid divides the line segment between orthocentre and circumcenter in the ratio 2 : 1

$$\Rightarrow \frac{H+2O}{3} = G$$

or H = 3G – 2O

$$G = \frac{A+B+C}{3}$$

$\Rightarrow$ H = A + B + C – 2O

$= (–a – x + x , b – y – y) = (–a, b – 2y)$

$\overline{MH} = \sqrt{a^2+(b-2y+r)^2} = r\text{ (given)}$

$\Rightarrow a^2 + b^2 + 4y^2 + r^2 – 4by – 4ry + 2br = r^2$

$\Rightarrow r^2 + 4y^2 – 4yr = 4yb – 2br$ … (1)

$2\angle BAC = \angle BOC = \angle BOM + \angle MOC$

$= 2\angle MOC$

$\Rightarrow \angle BAC = \angle MOC$

$\Rightarrow \cos(\angle BAC) = \text{COS } \angle MOC) = \frac{y}{r}$

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from (1)

$(r^2 - 4yr + 4y^2) = 2b(2y - r)$

$\Rightarrow (2y - r)^2 - (2y - r)(2b)$

$\Rightarrow (2y - r)(2y - r - b)$

$\Rightarrow 2y = r$ or $b - 2y = -r$

Since orthocenter of an acute Δ lies inside the Δ it must also lie within the circumcircle

$\Rightarrow |b - 2y| < r$

$\Rightarrow 2y = r$ or $\frac{a}{r} = \frac{1}{2} = \cos(\angle BAC)$

$\Rightarrow \angle BAC = 60°$ (ΔABC is acute)

6. Let p(x) be a nonconstant polynomial with integer coefficients, and let $n \geq 2$ be an integer such that no term of the sequence

P(0), p(p(0)), p(p(p(0))), ….

is divisible by n. Show that there exist integers a, b such that $0 \leq a < b \leq n - 1$ and n divides $p(b) - p(a)$.

Sol. FTSOC assume no integers a, b

$0 \leq a < b \leq n - 1$ exist such that $n \mid p(b) - p(a)$

$\Rightarrow \{p(0), p(1) .. p(n-1)\}$ forms a complete residue system.

A complete residue system mod n is an unordered set of integers where no 2 numbers are equal to each other mod n and there exists an integer i in the set such that

$i = j \pmod{n}\ \forall j \in \{0, 1, 2, \ldots n - 1\}$.

Now p(0), p(p(0)), p(p(p(0)))…. are integers not divisible by n

Consider $g : \{0, 1, 2 \ldots n - 1\} \to \{0, 1, 2, \ldots n - 1\}$

Be a discrete valued function such that

$q(x) = p(x) \bmod n\ \forall x \in \{0, 1, 2, \ldots, n - 1\}$

Since $\{p(0), p(1), \ldots p(n - 1)\}$ forms a complete residue system, g(x) will be a one-one function.

Claim there exists $k \in \{1, 2, \ldots n\}$ such that $q^k(0) = 0$

If no such k exists then $q^1(0), q^2(0) \ldots q^n(0) \neq 0$

n pigeons and n – 1 pigeon holes $\Rightarrow$ at least one $k \in \{1, 2, \ldots n - 1\}$ exists each that

$q^{\ell}(0) = q^m(0)$ where $\ell < m$, $\ell, m \in \{1, 2, \ldots n - 1\}$

$\Rightarrow q^{\ell-1}(0) = q^{m-1}(0)$ ($\because$ q is one-one)

$\Rightarrow q^{\ell-2}(0) = q^{m-2}(0)$

$\Rightarrow 0 = q^{m-\ell}(0)$ contradiction

$q^k(0) = 0 = p(p \ldots p(0)\ldots) \bmod n$ contradiction

Thus integers a, b, $0 \leq a < b \leq n - 1$ exist such that $n \mid p(b) - p(a)$