SOLUTION

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

AMTI – NMTC - 25th October, 2025, JUNIOR FINAL

1. m > 0, n > 0 are integers. Find all pairs (m, n) such that $m^2 + 3n$ and $n^2 + 3m$ are both perfect squares simultaneously.

Sol. Let, $m^2 + 3n = p^2$

$n^2 + 3m = q^2$

Now, if we take $m^2 + 3n \geq (m + 2)^2$ & $n^2 + 3m \geq (n + 2)^2$

it can not both be true, because adding them will lead to a contradiction. So atleast one of the inequalities $m^2 + 3n < (m + 2)^2$ and $n^2 + 3m < (n + 2)^2$ is true.

WLOG, let $m^2 + 3n < (m + 2)^2$, then

$m^2 < m^2 + 3n < (m + 2)^2$ implies

$m^2 + 3n = (m + 1)^2$

$m^2 + 3n = m^2 + 2m + 1$

$3n = 2m + 1 = \text{odd}$

Now, let n = 2k + 1 then m = 3k + 1 for some non negative integer k, and

We are solving one of the expressions

$$\begin{aligned} n^2 + 3m &= (2k + 1)^2 + 3(3k + 1) \\ &= 4k^2 + 4k + 1 + 9k + 3 \\ &= 4k^2 + 13k + 4 \end{aligned}$$

We have for k > 5, $(2k + 3)^2 < 4k^2 + 13k + 4 < (2k + 4)^2$,

hence $n^2 + 3m$ can not be a perfect square. Thus we need to check only $k \in \{0, 1, 2, 3, 4, 5\}$ only k = 0 and k = 5 makes $n^2 + 3m$ a perfect square.

m = n = 1, m = 16, n = 11, m = 11, n = 16

$\boxed{(m,n) = (1,1),(16,11),(11,16)}$

Total 3 pairs are possible

2. PQR is a right triangle with QPR = 90°. I is the incenter of the ΔPQR. The incircle touches side PQ at M and side PR at N. A line is drawn through I to cut PQ at A and PR at B. Prove that $PA. PB \geq 4r^2$, where r is the inradius of the ΔPQR.

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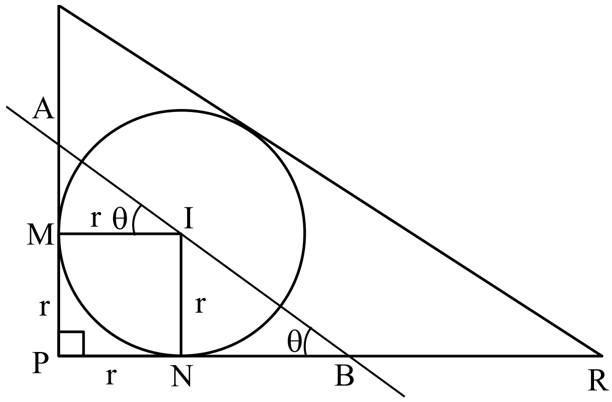
Sol.



From figure

$PA = MA + r$: $PB = r + NB$

Also $\Delta AMI \sim \Delta INB$ (By AA criteria)

$$\Rightarrow \frac{AM}{r} = \frac{r}{NB} \quad \Rightarrow \quad AM \times NB = r^2$$

From triangle AMI

$\Rightarrow$ $MA = r\tan\theta$

From triangle INB

$$\Rightarrow NB = \frac{r}{\tan\theta}$$

Now, $PA \times PB = (MA + r)(NB + r)$

$= MA \times NB + r(MA + NB) + r^2$

$$= 2r^2 + r\left(r\tan\theta + \frac{r}{\tan\theta}\right)$$

$$= 2r^2 + r^2\left(\tan\theta + \frac{1}{\tan\theta}\right)$$

As, $\tan\theta + \dfrac{1}{\tan\theta} \geq 2$ (AM $\geq$ GM)

$$\boxed{PA \times PB \geq 4r^2}$$

3. Solve for x, y, z from the following simultaneous equations :

$x^2 + xy + xz - x = 7$

$y^2 + yz + yx - y = -14$

$z^2 + zx + zy - z = 49$

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Sol. $x^2 + xy + xz - x = 7$

$y^2 + yz + yx - y = -14$

$z^2 + zx + zy - z = 49$

$$[x + y + z] = \frac{7 + x}{x} \qquad \ldots.(1)$$

$$[x + y + z] = \frac{-14 + y}{y} \qquad \ldots.(2)$$

$$[x + y + z] = \frac{49}{z} \qquad \ldots.(3)$$

As, we have

$$\underset{\text{I}}{\frac{x + 7}{x}} = \underset{\text{II}}{\frac{y - 14}{y}} = \underset{\text{III}}{\frac{z + 49}{z}} = K \text{ (let)}$$

$$\boxed{x = \frac{7}{K - 1}} \; \boxed{y = \frac{-14}{K - 1}} \; \boxed{z = \frac{49}{K - 1}}$$

From equation (1)

$$\left(\frac{7}{K-1} + \frac{-14}{K-1} + \frac{49}{K-1}\right)\left(\frac{7}{K-1}\right) = 7 + \frac{7}{K-1}$$

$$\frac{42}{(K-1)} \times \frac{7}{(K-1)} = \frac{7K - 7 + 7}{(K-1)}$$

$$\left(\frac{42}{K-1}\right)7 = 7K$$

$\Rightarrow K^2 - K - 42 = 0 \quad \Rightarrow \quad (K - 7)(K + 6) = 0$

$K = 7, -6$

If $K = 7$

$$\boxed{x = \frac{7}{6}}, \; \boxed{y = \frac{-14}{6} = \frac{-7}{3}}, \; \boxed{z = \frac{49}{6}}$$

If $K = \ 6$

$$\boxed{x = -1} \; \boxed{y = +2} \; \boxed{z = -7}$$

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4. Twenty-eight integers are chose from the interval [104, 208]. Prove that there exist two of them with a common prime divisor.

Sol. In [104, 208], there are 105 integers. Let A be the subset of integers in the interval [104, 208] that are divisible by atleast one of the primes 2,3,5,7 and let B be the complement set of A. Lets count no. of elements in A. For any positive integer d, let N_d denote the set of integers in the given interval that are divisible by d and N_d the number of elements in n_d. Clearly

$A = N_2 \cup N_3 \cup N_5 \cup N_7$. Note that

$n_d = \left[\frac{208}{d}\right] - \left[\frac{103}{d}\right]$, where [.] represents GIF.

Now, $n_2 = \left[\frac{208}{2}\right] - \left[\frac{103}{2}\right] = 104 - 51 = 53$

$n_3 = \left[\frac{208}{3}\right] - \left[\frac{103}{3}\right] = 69 - 34 = 35$

$n_5 = \left[\frac{208}{5}\right] - \left[\frac{103}{5}\right] = 41 - 20 = 21$

$n_7 = \left[\frac{208}{7}\right] - \left[\frac{103}{7}\right] = 29 - 14 = 15$

Similarly, $n_6 = 17$, $n_{10} = 10$, $n_{14} = 7$, $n_{15} = 7$, $n_{21} = 5$, $n_{35} = 3$, $n_{105} = 1$, $n_{70} = 1$, $n_{42} = 2$, $n_{30} = 3$, $n_{20} = 0$.

By PIE, number of elements in set A is

$|A| = n_2 + n_3 + n_5 + n_7 - (n_6 + n_{10} + n_{14} + n_{15} + n_{21} + n_{35} + (n_{105} + n_{70} + n_{42} + n_{30}) - n_{210} = 82$

Thus B contains 105 – 82 = 23 elements. When we select 28 elements, atleast 5 should be from set A. Each of these 5 is divisible by atleast one of the four primes. Hence by Pegion hole principle, atleast two must have a common prime divisor among 2, 3, 5, 7.

5. In the parallelogram ABCD, the diagonal AC is perpendicular to the side AD. AH is drawn perpendicular to CD. The tangent at D to the circumcircle of ΔADB meets CA produced at P. Prove that : ∠PBA = ∠DBH.

Sol. *

6. There are 6 friends, Manish, Cathy, Sundar, Elisha, Lakshaman and Divya. Manish told that the sum of our ages put together is five times my age. He again told that when Sundar will be three times my present age, the sum of my age and Divya's age will be equal to the sum of the present ages of five of us; Elisha's age will be three times her present age and Lakshaman's age will be twice Sundar's present age plus one year. They all started working to find their present ages and Cathy asked Manish that could he tell something more. Then Manish said that his age is an odd number. How old are Manish and Sundar?

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Sol. $M + C + S + E + L + D = 5M$

$C + S + E + L + D = 4M$ ....(i)

Let $t = 3M - S$ ....(iii)

$(M + t) + (D + t) = C + M + E + L + D$

$2t = C + E + L$ ....(ii)

$E + t = 3E \Rightarrow E = \frac{t}{2}$ ....(iv)

$L + t = 2S + 1 \Rightarrow L = 2S + 1 - t$ .....(v)

Eliminate C, E, L, D in term of M, S

$E = \frac{3M - S}{2}$, $L = 3S + 1 - 3M$

from (ii), $C = 2t - E - L$

$= 2(3M - S) - \left(\frac{3M - S}{2}\right) - (3S + 1 - 3M)$

$C = \frac{15M - 9S - 2}{2}$

from (i) & (ii); $D = 4M - 2t - S$

$D = 4M - 2(3M - S) - S = S - 2M$,

So all unknown are expressed in M & S

Now, $E > 0, \Rightarrow 3M > S$

$D > 0 \Rightarrow S > 2M$

$\therefore \boxed{2M < S < 3M}$

$C > 0 \Rightarrow 15M - 9S - 2 > 0$

So, $S < \frac{15M - 2}{9}$

But for any positive integer m,

$2M < \frac{15M - 2}{9}$

$18M < 15M - 2$

$3M < -2$

$M < \frac{-2}{3}$ (not possible)

$\therefore$ both inequalities cannot be hold for any positive M.

So there is no possible solution to given problem.

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7. Solve in positive integers : $\dfrac{x^2 - y}{34x - y^2} = \dfrac{y}{x}$, provided $34x - y^2 \neq 0$.

Sol. $\dfrac{x^2 - y}{34x - y^2} = \dfrac{y}{x}$

$x^3 - xy = 34xy - y^3$

$x^3 + y^3 = 35xy$

let $x = ad$, $y = bd$ where

$d = \gcd(x, y)$ and $\gcd(a, b) = 1$

$\Rightarrow d^3(a^3 + b^3) = 35d^2ab$

$\Rightarrow d(a^3 + b^3) = 35\,ab$

Since a & b both are co-prime to $a^3 + b^3$

$\therefore\ a^3 + b^3 = 35$ or factors of 35.

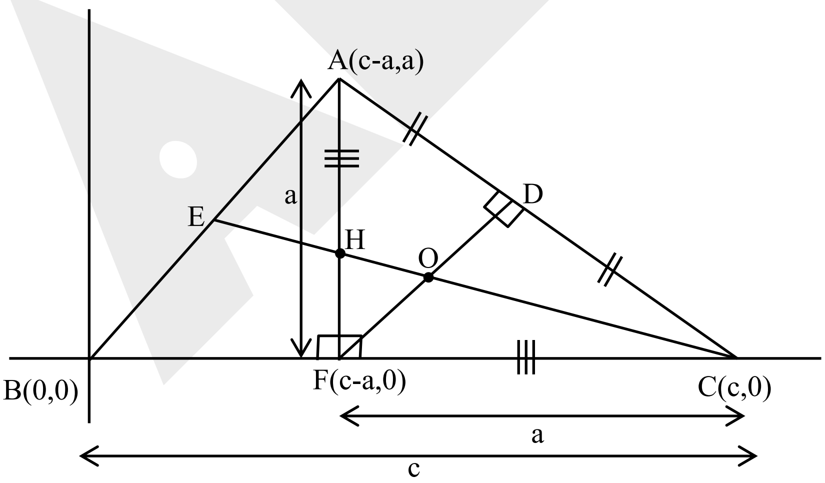
By observation.

This is true only for $(a, b) = (2, 3)$

hence $d = 6$ and $(x, y) = (12, 18), (18, 12)$

8. Let ABC be an acute-angled triangle. Let points O, H be its circumcenter and orthocenter respectively. The altitude from A, the perpendicular bisector of side AC and side BC are concurrent. Then find CH:BO.

Sol. Let us assume an acute angled Δ on x-axis.



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H ≡ Ortho-center

O ≡ Circum-center

Now, we have to find co-ordinate of 'H' & 'O'.

(a) For coordinates of $\boxed{H}$.

Let H (c-a, y)

We can find value of y solving perpendicular line drawn from c to AB i.e. CE and another line AF.

$\Rightarrow$ Equation of CE $\Rightarrow$ $\boxed{y=\left(\frac{a-c}{a}\right)(x-c)}$ ....(1)

$\Rightarrow$ Equation of AF $\Rightarrow$ $\boxed{x=c-a}$ ....(2)

By solve Eq^n (1) & (2).

$\boxed{y=c-a}$

$\Rightarrow$ Co-ordinate of H ≡ (c–a, c–a)

(b) For finding $\boxed{O}$ coordinates

$\Rightarrow$ $O \equiv \left(\frac{c}{2}, k\right)$

Apply $\boxed{OA=OB}$

$(OA)^2 = (OB)^2$

$\Rightarrow$ $\left[\frac{c}{2}-(c-a)\right]^2 + (k-a)^2 = \left(\frac{c}{2}\right)^2 + k^2$

By solving $k = \left(\frac{2a-c}{2}\right)$

Then co-ordinate of $O \equiv \left(\frac{c}{2}, \frac{2a-c}{2}\right)$

Then, we have line segments

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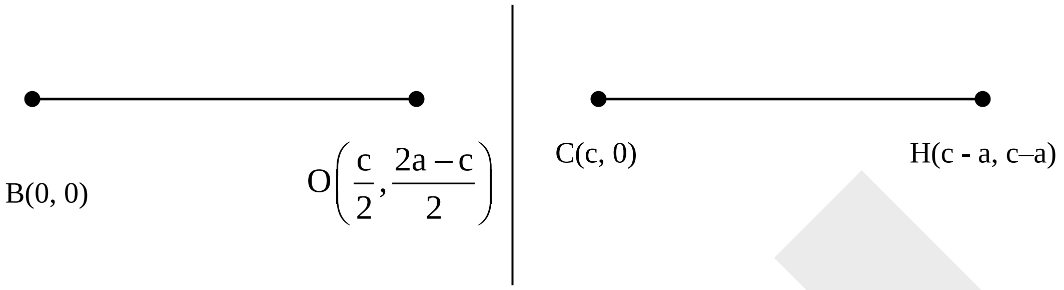

$$\frac{(CH)^2}{(BO)^2} = \frac{\left[\left((c-a)-c\right)^2 + (c-a)^2\right]}{\left[\left(\frac{c}{2}-0\right)^2 + \left(\frac{2a-c}{2}\right)^2\right]}$$

$$= \frac{a^2+c^2+a^2-2ac}{\left(\frac{c^2+4a^2+c^2-4ac}{4}\right)} = \frac{4}{2} = 2$$

$$\Rightarrow \boxed{\frac{CH}{BO} = \frac{\sqrt{2}}{1}}$$