

FINAL JEE-MAIN EXAMINATION – JUNE, 2022

(Held On Saturday 25th June, 2022)

TIME : 9 : 00 AM to 12 : 00 PM

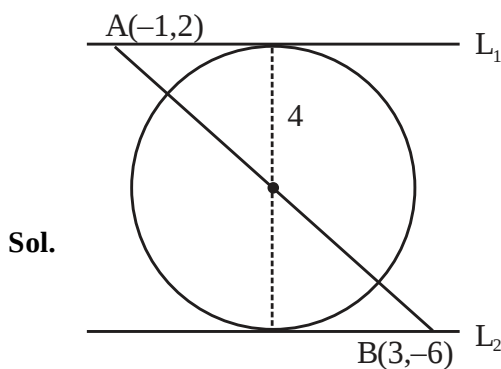
MATHEMATICS

SECTION-A

1. Let a circle C touch the lines $L_1 : 4x - 3y + K_1 = 0$ and $L_2 : 4x - 3y + K_2 = 0$, $K_1, K_2 \in \mathbb{R}$. If a line passing through the centre of the circle C intersects L_1 at $(-1, 2)$ and L_2 at $(3, -6)$, then the equation of the circle C is
- (A) $(x - 1)^2 + (y - 2)^2 = 4$
 (B) $(x + 1)^2 + (y - 2)^2 = 4$
 (C) $(x - 1)^2 + (y + 2)^2 = 16$
 (D) $(x - 1)^2 + (y - 2)^2 = 16$

Official Ans. by NTA (C)

Allen Ans. (C)



$$L_1 : 4x - 3y + K_1 = 0$$

$$L_2 : 4x - 3y + K_2 = 0$$

now

$$-4 - 6 + K_1 = 0 \Rightarrow K_1 = 10$$

$$12 + 18 + K_2 = 0 \Rightarrow K_2 = -30$$

$\Rightarrow$ Tangent to the circle are

$$4x - 3y + 10 = 0$$

$$4x - 3y - 30 = 0$$

$$\text{Length of diameter } 2r = \frac{|10 + 30|}{5} = 8$$

$$\Rightarrow r = 4$$

Now centre is mid point of A & B

$$x = 1, y = -2$$

Equation of circle

$$(x - 1)^2 + (y + 2)^2 = 16 \text{ Ans.}$$

TEST PAPER WITH SOLUTION

2. The value of $\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx$ is equal to
- (A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^2}{2}$
 (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx \dots (1)$

Use King's property

$$I = \int_0^{\pi} \frac{e^{-\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx \dots (2)$$

On adding equation (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx = 2 \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$$

On putting $\cos x = t$, we get

$$I = \int_0^1 \frac{dt}{1 + t^2} = (\tan^{-1} t)_0^1 = \frac{\pi}{4}$$

3. Let a, b and c be the length of sides of a triangle

ABC such that $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9}$. If r and R are the radius of incircle and radius of circumcircle of the triangle ABC, respectively,

then the value of $\frac{R}{r}$ is equal to

- (A) $\frac{5}{2}$ (B) 2
 (C) $\frac{3}{2}$ (D) 1

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9} = \lambda$

$a + b = 7\lambda, b + c = 8\lambda, a + c = 9\lambda$

$\Rightarrow a + b + c = 12\lambda$

Now $a = 4\lambda, b = 3\lambda, c = 5\lambda$

$\therefore c^2 = b^2 + a^2$

$\angle C = 90^\circ$

$\Delta = \frac{1}{2}ab \sin C = \frac{1}{2}ab$

$\frac{R}{r} = \frac{c}{2 \sin C} \times \frac{s}{\Delta} = \frac{c}{2} \times \frac{6\lambda}{\frac{1}{2}ab} = \frac{c}{ab} \times 6\lambda = \frac{5}{2}$

4. Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a function such that $f(x+y) = 2f(x)f(y)$ for natural numbers x and y . If $f(1) = 2$, then the value of α for which

$\sum_{k=1}^{10} f(\alpha + k) = \frac{512}{3}(2^{20} - 1)$

holds, is

- (A) 2 (B) 3
(C) 4 (D) 6

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $f : \mathbb{N} \rightarrow \mathbb{R}, f(x+y) = 2f(x)f(y) \dots (1)$
 $f(1) = 2,$

$\sum_{k=1}^{10} f(\alpha + k) = 2f(\alpha) \sum_{k=1}^{10} f(k)$
 $= 2f(\alpha)(f(1) + f(2) + \dots + f(10)) \dots (2)$

From (1)

$f(2) = 2f(1) = 2^3$

$f(3) = 2f(2)f(1) = 2^5$

$\vdots$

$f(10) = 2^9 f(1) = 2^{19}$

$f(\alpha) = 2^{2\alpha-1}; \alpha \in \mathbb{N}$

from (2)

$\sum_{k=1}^{10} f(\alpha + k) = 2(2^{2\alpha-1})(2 + 2^3 + 2^5 + \dots + 2^{19})$

$\frac{512}{3}(2^{20} - 1) = 2^{2\alpha} \left(2 \frac{(2^{20} - 1)}{3} \right)$

Hence $\alpha = 4$

5. Let A be a 3×3 real matrix such that

$A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$

If $X = (x_1, x_2, x_3)^T$ and I is an identity matrix

of order 3, then the system $(A - 2I)X = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$

has

- (A) no solution
(B) infinitely many solutions
(C) unique solution
(D) exactly two solutions

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\Rightarrow c_1 = 1, c_2 = 1, c_3 = 2$

$A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + a_1 \\ c_2 + a_2 \\ c_3 + a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow a_1 = -2, a_2 = -1, a_3 = -1$

$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow b_1 = 3, b_2 = 2, b_3 = 1$

$\Rightarrow A = \begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

$\Rightarrow A - 2I = \begin{bmatrix} -4 & 3 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$|A - 2I| = 0$

Now, $\begin{bmatrix} -4 & 3 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

$-4x_1 + 3x_2 + x_3 = 4 \dots (1)$

$-x_1 + x_3 = 1 \dots (2)$

$-x_1 + x_2 = 1 \dots (3)$

$(1) - [(2) + 3(3)]$

$0 = 0 \Rightarrow$ infinite solutions

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x^3 + x - 5$. If $g(x)$ is a function such that $f(g(x)) = x$, $\forall x \in \mathbb{R}$, then $g'(63)$ is equal to _____.

- (A) $\frac{1}{49}$ (B) $\frac{3}{49}$
(C) $\frac{43}{49}$ (D) $\frac{91}{49}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $f(x) = x^3 + x - 5$
 $\Rightarrow f'(x) = 3x^2 + 1 \Rightarrow$ increasing function
 $\Rightarrow$ invertible
 $\Rightarrow g(x)$ is inverse of $f(x)$
 $\Rightarrow g(f(x)) = x$
 $\Rightarrow g'(f(x))f'(x) = 1$
 $f(x) = 63$
 $\Rightarrow x^3 + x - 5 = 63$
 $\Rightarrow x = 4$
 put $x = 4$
 $g'(f(4))f'(4) = 1$
 $g'(63) \times 49 = 1 \quad \{f(4) = 49\}$
 $g'(63) = \frac{1}{49}$

7. Consider the following two propositions:

$P_1 : \sim(p \rightarrow \sim q)$

$P_2 : (p \wedge \sim q) \wedge ((\sim p) \vee q)$

If the proposition $p \rightarrow ((\sim p) \vee q)$ is evaluated as FALSE, then:

- (A) P_1 is TRUE and P_2 is FALSE
 (B) P_1 is FALSE and P_2 is TRUE
 (C) Both P_1 and P_2 are FALSE
 (D) Both P_1 and P_2 are TRUE

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \rightarrow (\sim p \vee q)$	$p \rightarrow \sim q$	$\sim(p \rightarrow \sim q)$	$p \wedge \sim q$	p_2
T	T	F	F	T	T	F	T	F	F
T	F	F	T	F	F	T	F	T	F
F	T	T	F	T	T	F	T	F	F
F	F	T	T	T	T	T	F	F	F

$p \rightarrow (\sim p \vee q)$ is F when p is true q is false

From table

P_1 & P_2 both are false

8. If $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \dots + \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$, then the remainder when K is divided by 6 is
 (A) 1 (B) 2
 (C) 3 (D) 5

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \frac{1}{2^3 \cdot 3^8} + \dots + \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$

$$K = 2^9 + 2^8 \cdot 3 + 2^7 \cdot 3^2 + \dots + 3^9$$

$$= \frac{2^9 \left(\left(\frac{3}{2} \right)^{10} - 1 \right)}{\frac{3}{2} - 1} = 3^{10} - 2^{10}$$

$$\begin{aligned} \text{Now, } 3^{10} - 2^{10} &= (3^5 - 2^5)(3^5 + 2^5) \\ &= (211)(275) \\ &= (35 \times 6 + 1)(45 \times 6 + 5) \\ &= 6\lambda + 5 \end{aligned}$$

Remainder is 5.

9. Let $f(x)$ be a polynomial function such that $f(x) + f'(x) + f''(x) = x^5 + 64$. Then, the value of $\lim_{x \rightarrow 1} \frac{f(x)}{x-1}$

- (A) -15 (B) -60
(C) 60 (D) 15

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\lim_{x \rightarrow 1} \frac{f(x)}{x-1} = f'(1)$ (and $f(1) = 0$)

$$f(x) + f'(x) + f''(x) = x^5 + 64$$

$$f'(x) + f''(x) + f'''(x) = 5x^4$$

$$f''(x) + f'''(x) + f^{iv}(x) = 20x^3$$

$$f'''(x) + f^{iv}(x) + f^v(x) = 60x^2$$

$$\therefore f^v(x) - f''(x) = 60x^2 - 20x^3$$

$$\Rightarrow 120 - f''(1) = 40 \Rightarrow f''(1) = 80$$

$$\text{Also } f(1) + f'(1) + f''(1) = 65 \Rightarrow f'(1) = -15. \text{ Ans.}$$

10. Let E_1 and E_2 be two events such that the conditional probabilities $P(E_1|E_2) = \frac{1}{2}$,

$P(E_2|E_1) = \frac{3}{4}$ and $P(E_1 \cap E_2) = \frac{1}{8}$. Then:

- (A) $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$
 (B) $P(E'_1 \cap E'_2) = P(E'_1) \cdot P(E'_2)$
 (C) $P(E_1 \cap E'_2) = P(E_1) \cdot P(E_2)$
 (D) $P(E'_1 \cap E_2) = P(E_1) \cdot P(E_2)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.

(A) $P(E_1) \cdot P(E_2) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24} \neq P(E_1 \cap E_2)$

(B) $P(E'_1 \cap E'_2) = 1 - P(E_1 \cup E_2)$
 $= 1 - (P(E_1) + P(E_2) - P(E_1 \cap E_2))$
 $= 1 - \left(\frac{1}{6} + \frac{1}{4} - \frac{1}{8} \right) = \frac{17}{24}$

$P(E'_1)P(E_2) = \frac{5}{6} \times \frac{1}{4} = \frac{5}{24}$

(C) $P(E_1 \cap E'_2) = P(E_1) - P(E_1 \cap E_2) = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$

(D) $P(E'_1 \cap E_2) = P(E_2) - P(E_1 \cap E_2) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$

11. Let $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. If M and N are two matrices

given by $M = \sum_{k=1}^{10} A^{2k}$ and $N = \sum_{k=1}^{10} A^{2k-1}$ then

MN^2 is

- (A) a non-identity symmetric matrix
 (B) a skew-symmetric matrix
 (C) neither symmetric nor skew-symmetric matrix
 (D) an identity matrix

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = -4I$

$A^3 = -4A$

$A^4 = (-4I)(-4I) = (-4)^2 I$

$A^5 = (-4)^2 A, A^6 = (-4)^3 I$

$M = \sum_{k=1}^{10} A^{2k} = A^2 + A^4 + \dots + A^{20}$

$= [-4 + (-4)^2 + (-4)^3 + \dots + (-4)^{20}] I$

$= -4\lambda I$

$\Rightarrow M$ is symmetric matrix

$N = \sum_{k=1}^{10} A^{2k-1} = A + A^3 + \dots + A^{19}$

$= A[1 + (-4) + (-4)^2 + \dots + (-4)^9]$

$= \lambda A \Rightarrow$ skew symmetric

$\Rightarrow N^2$ is symmetric matrix

$\Rightarrow MN^2$ is non identity symmetric matrix

12. Let $g : (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that

$\int \left(\frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c,$

for all $x > 0$, where c is an arbitrary constant. Then.

(A) g is decreasing in $\left(0, \frac{\pi}{4}\right)$

(B) g' is increasing in $\left(0, \frac{\pi}{4}\right)$

(C) $g + g'$ is increasing in $\left(0, \frac{\pi}{2}\right)$

(D) $g - g'$ is increasing in $\left(0, \frac{\pi}{2}\right)$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.

$$\int \left(\frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c$$

On differentiating both sides w.r.t. x , we get

$$\left(\frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right)$$

$$= \frac{(e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)}{(e^x + 1)^2}$$

$$(e^x + 1)x(\cos x - \sin x) + g(x)(e^x + 1 - xe^x)$$

$$= (e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)$$

$$\Rightarrow g'(x) = \cos x - \sin x$$

$$\Rightarrow g(x) = \sin x + \cos x + C$$

$$g(x) \text{ is increasing in } (0, \pi/4)$$

$$g''(x) = -\sin x - \cos x < 0$$

$$\Rightarrow g'(x) \text{ is decreasing function}$$

$$\text{let } h(x) = g(x) + g'(x) = 2 \cos x + C$$

$$\Rightarrow h'(x) = g'(x) + g''(x) = -2 \sin x < 0$$

$$\Rightarrow h \text{ is decreasing}$$

$$\text{let } \phi(x) = g(x) - g'(x) = 2 \sin x + C$$

$$\Rightarrow \phi'(x) = g'(x) - g''(x) = 2 \cos x > 0$$

$$\Rightarrow \phi \text{ is increasing}$$

Hence option D is correct.

13. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions defined by $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$ and

$g(x) = \frac{1 - 2e^{2x}}{e^x}$. Then, for which of the following range of α , the inequality

$$f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right) \text{ holds?}$$

$$(A) (2, 3) \quad (B) (-2, -1)$$

$$(C) (1, 2) \quad (D) (-1, 1)$$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$

$$\Rightarrow f'(x) = \frac{2x}{x^2 + 1} + e^{-x} > 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow f \text{ is strictly increasing}$$

$$g(x) = \frac{1 - 2e^{2x}}{e^x} = e^{-x} - 2e^x$$

$$\Rightarrow g'(x) = -(2e^x + e^{-x}) < 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow g \text{ is decreasing}$$

$$\text{Now } f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right)$$

$$\Rightarrow g\left(\frac{(\alpha-1)^2}{3}\right) > g\left(\alpha - \frac{5}{3}\right)$$

$$\Rightarrow \frac{(\alpha-1)^2}{3} < \alpha - \frac{5}{3}$$

$$\Rightarrow \alpha^2 - 5\alpha + 6 < 0$$

$$\Rightarrow (\alpha - 2)(\alpha - 3) < 0$$

$$\Rightarrow \alpha \in (2, 3)$$

14. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ $a_i > 0$, $i = 1, 2, 3$ be a vector which makes equal angles with the coordinates axes OX, OY and OZ. Also, let the projection of $\vec{a}$ on the vector $3\hat{i} + 4\hat{j}$ be 7. Let $\vec{b}$ be a vector obtained by rotating $\vec{a}$ with 90° .

If $\vec{a}$, $\vec{b}$ and x-axis are coplanar, then projection of a vector $\vec{b}$ on $3\hat{i} + 4\hat{j}$ is equal to

$$(A) \sqrt{7}$$

$$(B) \sqrt{2}$$

$$(C) 2$$

$$(D) 7$$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a} = \lambda \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right) = \frac{\lambda}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

Now projection of $\vec{a}$ on $\vec{b} = 7$

$$\Rightarrow \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 7$$

$$\frac{\lambda}{\sqrt{3}} \frac{(\hat{i} + \hat{j} + \hat{k}) \cdot (3\hat{i} + 4\hat{j})}{5} = 7$$

$$\lambda = 5\sqrt{3}$$

$$\vec{a} = 5(\hat{i} + \hat{j} + \hat{k})$$

$$\text{now } \vec{b} = 5\alpha(\hat{i} + \hat{j} + \hat{k}) + \beta(\hat{i})$$

$$\vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow 25\alpha(3) + 5\beta = 0$$

$$\Rightarrow 15\alpha + \beta = 0 \Rightarrow \beta = -15\alpha$$

$$\vec{b} = 5\alpha(-2\hat{i} + \hat{j} + \hat{k})$$

$$|\vec{b}| = 5\sqrt{3}$$

$$\Rightarrow \alpha = \pm \frac{1}{\sqrt{2}}$$

$$\vec{b} = \pm \frac{5}{\sqrt{2}}(-2\hat{i} + \hat{j} + \hat{k})$$

Projection of $\vec{b}$ on $3\hat{i} + 4\hat{j}$ is

$$\frac{\vec{b} \cdot (3\hat{i} + 4\hat{j})}{5} = \pm \frac{5}{\sqrt{2}} \left(\frac{-6+4}{5} \right) = \pm \sqrt{2}$$

15. Let $y = y(x)$ be the solution of the differential equation $(x+1)y' - y = e^{3x}(x+1)^2$, with

$$y(0) = \frac{1}{3}. \text{ Then, the point } x = -\frac{4}{3} \text{ for the curve}$$

$y = y(x)$ is:

- (A) not a critical point
- (B) a point of local minima
- (C) a point of local maxima
- (D) a point of inflection

Official Ans. by NTA (B)

Allen Ans. (B)

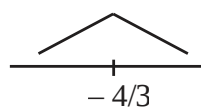
Sol. $(x+1)dy - y dx = e^{3x}(x+1)^2$

$$\frac{(x+1)dy - ydx}{(x+1)^2} = e^{3x}$$

$$d\left(\frac{y}{x+1}\right) = e^{3x} \Rightarrow \frac{y}{x+1} = \frac{e^{3x}}{3} + C$$

$$\left(0, \frac{1}{3}\right) \Rightarrow C = 0 \Rightarrow y = \frac{(x+1)e^{3x}}{3}$$

$$\frac{dy}{dx} = \frac{1}{3}((x+1)3e^{3x} + e^{3x}) = \frac{3^{3x}}{3}(3x+4)$$



Clearly, $x = -\frac{4}{3}$ is point of local minima

16. If $y = m_1x + c_1$ and $y = m_2x + c_2$, $m_1 \neq m_2$ are two common tangents of circle $x^2 + y^2 = 2$ and parabola $y^2 = x$, then the value of $8|m_1m_2|$ is equal to

(A) $3+4\sqrt{2}$ (B) $-5+6\sqrt{2}$

(C) $-4+3\sqrt{2}$ (D) $7+6\sqrt{2}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $C_1: x^2 + y^2 = 2$

$$C_2: y^2 = x$$

Let tangent to parabola be $y = mx + \frac{1}{4m}$.

It is also a tangent of circle so distance from centre of circle (0, 0) will be $\sqrt{2}$.

$$\left| \frac{\frac{1}{4m}}{\sqrt{1+m^2}} \right| = \sqrt{2} \Rightarrow 1 = 32m^2 + 32m^4$$

by solving

$$m^2 = \frac{3\sqrt{2}-4}{8}, m^2 = \frac{-3\sqrt{2}-4}{8} \text{ (rejected)}$$

$$m = \pm \sqrt{\frac{3\sqrt{2}-4}{8}}$$

$$\text{so, } 8|m_1m_2| = 3\sqrt{2}-4$$

17. Let Q be the mirror image of the point P(1, 0, 1) with respect to the plane S : $x + y + z = 5$. If a line L passing through (1, -1, -1), parallel to the line PQ meets the plane S at R, then QR^2 is equal to:

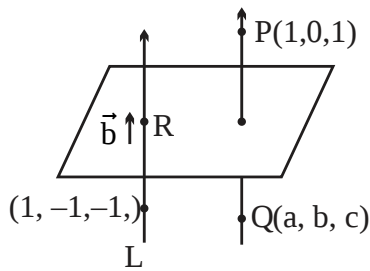
(A) 2 (B) 5

(C) 7 (D) 11

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



Let parallel vector of L = $\vec{b}$

mirror image of Q on given plane $x+y+z=5$

$$\frac{a-1}{1} = \frac{b-0}{1} = \frac{c-1}{1} = \frac{-2(2-5)}{3}$$

$$a = 3, b = 2, c = 3$$

$$Q \equiv (3, 2, 3)$$

$$\therefore \vec{b} \parallel \overrightarrow{PQ}$$

$$\text{so, } \vec{b} = (1, 1, 1)$$

Equation of line

$$L : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z+1}{1}$$

Let point R, $(\lambda+1, \lambda-1, \lambda-1)$

lying on plane $x + y + z = 5$,

$$\text{so, } 3\lambda - 1 = 5$$

$$\Rightarrow \lambda = 2$$

Point R is $(3, 1, 1)$

$$QR^2 = 5 \text{ Ans.}$$

18. If the solution curve $y = y(x)$ of the differential equation $y^2 dx + (x^2 - xy + y^2) dy = 0$, which passes through the point $(1, 1)$ and intersects the line $y = \sqrt{3}x$ at the point $(\alpha, \sqrt{3}\alpha)$, then value of $\log_e(\sqrt{3}\alpha)$ is equal to

- (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$
(C) $\frac{\pi}{12}$ (D) $\frac{\pi}{6}$

Official Ans. by NTA (C)

Allen Ans. (C)

$$\begin{aligned} \text{Sol. } y^2 dx - xy dy &= -(x^2 + y^2) dy \\ y(y dx - x dy) &= -(x^2 + y^2) dy \\ -y(x dx - y dy) &= -(x^2 + y^2) dy \end{aligned}$$

$$\frac{xdy - ydx}{x^2} = \left(1 + \frac{y^2}{x^2}\right) \frac{dy}{y}$$

$$\Rightarrow \frac{d(y/x)}{1 + \frac{y^2}{x^2}} = \frac{dy}{y}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln y + C$$

$$(\alpha, \sqrt{3}\alpha) \Rightarrow \frac{\pi}{3} = \ln(\sqrt{3}\alpha) + \frac{\pi}{4}$$

$$\therefore \ln(\sqrt{3}\alpha) = \frac{\pi}{12}$$

19. Let $x = 2t$, $y = \frac{t^2}{3}$ be a conic. Let S be the focus

and B be the point on the axis of the conic such that $SA \perp BA$, where A is any point on the conic. If k is the ordinate of the centroid of

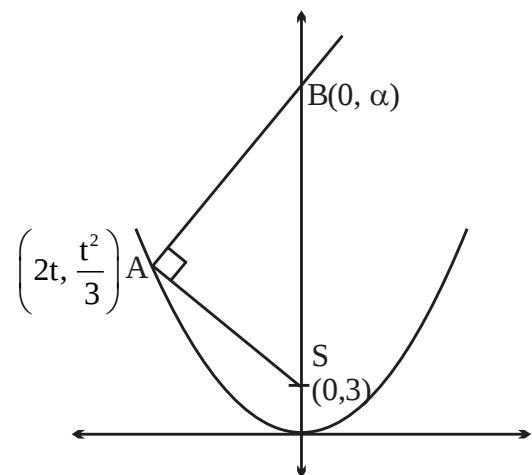
ΔSAB , then $\lim_{t \rightarrow 1} k$ is equal to

- (A) $\frac{17}{18}$ (B) $\frac{19}{18}$
(C) $\frac{11}{18}$ (D) $\frac{13}{18}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.



parabola $x^2 = 12y$

$SA \perp SB$

$$\text{so, } m_{AS} \cdot m_{AB} = -1$$

$$\frac{\left(3 - \frac{t^2}{3}\right)}{(0-2t)} \cdot \frac{\left(\alpha - \frac{t^2}{3}\right)}{(0-2t)} = -1$$

by solving

$$3\alpha = \frac{27t^2 + t^4}{t^2 - 9}$$

$$\text{ordinate of centroid of } \Delta SAB = K = \frac{\alpha + \frac{t^2}{3} + 3}{3}$$

$$k = \frac{9 + 3\alpha + t^2}{9}$$

$$\lim_{t \rightarrow 1} k = \lim_{t \rightarrow 1} \frac{1}{9} \left(9 + t^2 + \frac{27t^2 + t^4}{(t^2 - 9)} \right) = \frac{13}{18}$$

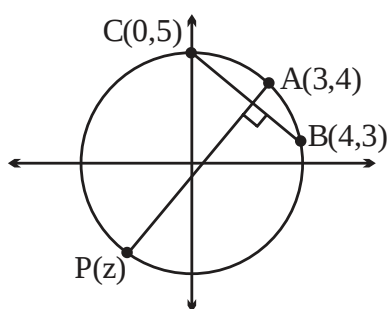
20. Let a circle C in complex plane pass through the points $z_1 = 3 + 4i$, $z_2 = 4 + 3i$ and $z_3 = 5i$. If $z (\neq z_1)$ is a point on C such that the line through z and z_1 is perpendicular to the line through z_2 and z_3 , then $\arg(z)$ is equal to :

(A) $\tan^{-1}\left(\frac{2}{\sqrt{5}}\right) - \pi$ (B) $\tan^{-1}\left(\frac{24}{7}\right) - \pi$

(C) $\tan^{-1}(3) - \pi$ (D) $\tan^{-1}\left(\frac{3}{4}\right) - \pi$

Official Ans. by NTA (B)

Allen Ans. (B)



Sol.

$$\text{Slope of BC} = \frac{3-5}{4-0} = -\frac{1}{2}$$

$$\text{Slope of AP} = 2$$

$$\text{equation of AP : } y - 4 = 2(x - 3)$$

$$\Rightarrow y = 2(x - 1)$$

$$P \text{ lies on circle } x^2 + y^2 = 25$$

$$\Rightarrow x^2 + (2(x - 1))^2 = 25$$

$$\Rightarrow x = -\frac{7}{5} \text{ and } y = -\frac{24}{5}$$

$$\Rightarrow \arg(z) = \tan^{-1}\left(\frac{24}{7}\right) - \pi$$

SECTION-B

1. Let C_r denote the binomial coefficient of x^r in the expansion of $(1+x)^{10}$. If $\alpha, \beta \in \mathbb{R}$. $C_1 + 3 \cdot 2C_2 + 5 \cdot 3C_3 + \dots$ upto 10 terms

$$= \frac{\alpha \times 2^{11}}{2^\beta - 1} \left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots \text{upto 10 terms} \right)$$

then the value of $\alpha + \beta$ is equal to

Official Ans. by NTA (286)

Allen Ans. (BONUS)

$$\text{Sol. } (1+x)^{10} = C_0 + C_1x + C_2x^2 + \dots + C_{10}x^{10}$$

Differentiating

$$10(1+x)^9 = C_1 + 2C_2x + 3C_3x^2 + \dots + 10C_{10}x^9$$

replace $x \rightarrow x^2$

$$10(1+x^2)^9 = C_1 + 2C_2x^2 + 3C_3x^4 + \dots + 10C_{10}x^{18}$$

$$10 \cdot x(1+x^2)^9 = C_1x + 2C_2x^3 + 3C_3x^5 + \dots + 10C_{10}x^{19}$$

Differentiating

$$10 \left((1+x^2)^9 \cdot 1 + x \cdot 9(1+x^2)^8 \cdot 2x \right)$$

$$= C_1x + 2C_2 \cdot 3x^3 + 3 \cdot 5 \cdot C_3x^4 + \dots + 10 \cdot 19C_{10}x^{18}$$

putting $x = 1$

$$10(2^9 + 18 \cdot 2^8)$$

$$= C_1 + 3 \cdot 2 \cdot C_2 + 5 \cdot 3 \cdot C_3 + \dots + 19 \cdot 10 \cdot C_{10}$$

$$C_1 + 3 \cdot 2 \cdot C_2 + \dots + 19 \cdot 10 \cdot C_{10}$$

$$= 10 \cdot 2^9 \cdot 10 = 100 \cdot 2^9$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_9}{11} + \frac{C_{10}}{11} = \frac{2^{11} - 1}{11}$$

10th term 11th term

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_9}{11} = \frac{2^{11} - 2}{11}$$

$$\text{Now, } 100 \cdot 2^9 = \frac{\alpha \cdot 2^{11}}{2^\beta - 1} \left(\frac{2^{11} - 2}{11} \right)$$

Eqn. of form $y = k(2^x - 1)$.

It has infinite solutions even if we take $x, y \in \mathbb{N}$.

2. The number of 3-digit odd numbers, whose sum of digits is a multiple of 7, is _____.

Official Ans. by NTA (63)

Allen Ans. (63)

Sol. $x y z \leftarrow$ odd number

$$z = 1, 3, 5, 7, 9$$

$$x+y+z = 7, 14, 21 \text{ [sum of digit multiple of 7]}$$

$$\begin{matrix} x & + & y \\ 1 \text{ to } 9 & & 0 \text{ to } 9 \end{matrix} = 6, 4, 2, 13, 11, 9, 7, 5, 20, 18, 16, 14, 12$$

$$x + y = 6 \Rightarrow (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)$$

$$\rightarrow \text{T.N.} = 6$$

$$x + y = 4 \Rightarrow (1, 3), (2, 2), (3, 1), (4, 0)$$

$$\rightarrow \text{T.N.} = 4$$

$$x + y = 2 \Rightarrow (1, 1), (2, 0)$$

$$\rightarrow \text{T.N.} = 2$$

$$x + y = 13 \Rightarrow (4, 9), (5, 8), (6, 7), (7, 6), (8, 5), (9, 4)$$

$$\rightarrow \text{T.N.} = 6$$

$$x + y = 11 \Rightarrow (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (6, 5), (7, 4), (8, 3), (9, 2)$$

$$\rightarrow \text{T.N.} = 8$$

$$x + y = 9 \Rightarrow (1, 8), (2, 7), (3, 6), (4, 5), (5, 4), \dots, (8, 1), (9, 0)$$

$$\rightarrow \text{T.N.} = 9$$

$$x + y = 7 \Rightarrow (1, 6), (2, 5), (3, 4), \dots, (8, 1), (7, 0)$$

$$\rightarrow \text{T.N.} = 7$$

$$x + y = 5 \Rightarrow (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)$$

$$\rightarrow \text{T.N.} = 5$$

$$x + y = 20 \Rightarrow \text{Not possible}$$

$$x + y = 18 \Rightarrow (9, 9) \rightarrow \text{T.N.} = 1$$

$$x + y = 16 \Rightarrow (7, 9), (8, 8), (9, 7)$$

$$\rightarrow \text{T.N.} = 3$$

$$x + y = 14 \Rightarrow (5, 9), (6, 8), (7, 7), (8, 6), (9, 5)$$

$$\rightarrow \text{T.N.} = 5$$

$$x + y = 12 \Rightarrow (3, 9), (4, 8), (5, 7), (6, 6), \dots, (9, 3)$$

$$\rightarrow \text{T.N.} = 7$$

3. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$,

$$\text{where } |\vec{a}| = 4, |\vec{b}| = 3 \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{3} \right). \text{ Then}$$

$$\left| (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) \right|^2 + 4(\vec{a} \cdot \vec{b})^2 \text{ is equal to } \underline{\hspace{2cm}}$$

Official Ans. by NTA (576)

Allen Ans. (576)

$$\text{Sol. } |\vec{a}| = 4, |\vec{b}| = 3 \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{3} \right)$$

$$\left| (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) \right|^2 + 4(\vec{a} \cdot \vec{b})^2$$

$$\left| \vec{a} \times \vec{b} - \vec{b} \times \vec{a} \right|^2 + 4a^2b^2 \cos^2 \theta$$

$$2 \left| \vec{a} \times \vec{b} \right|^2 + 4a^2b^2 \cos^2 \theta$$

$$4a^2b^2 \sin^2 \theta + 4a^2b^2 \cos^2 \theta$$

$$4a^2b^2 = 4 \times 16 \times 9 = 576$$

4. Let the abscissae of the two points P and Q be the roots of $2x^2 - rx + p = 0$ and the ordinates of P and Q be the roots of $x^2 - sx - q = 0$. If the equation of the circle described on PQ as diameter is $2(x^2 + y^2) - 11x - 14y - 22 = 0$, then $2r + s - 2q + p$ is equal to

Official Ans. by NTA (7)

Allen Ans. (7)

$$\text{Sol. } 2x^2 - rx + p = 0 \begin{cases} x_1 \\ x_2 \end{cases}$$

$$y^2 - sy - q = 0 \begin{cases} y_1 \\ y_2 \end{cases}$$

$$\text{Equation of the circle with PQ as diameter is } 2(x^2 + y^2) - rx - 2sy + p - 2q = 0$$

$$\text{on comparing with the given equation}$$

$$r = 11, s = 7$$

$$p - 2q = -22$$

$$\therefore 2r + s - 2q + p = 22 + 7 - 22 = 7$$

5. The number of values of x in the interval

$$\left(\frac{\pi}{4}, \frac{7\pi}{4} \right) \text{ for which } 14 \operatorname{cosec}^2 x - 2 \sin^2 x = 21$$

$$- 4 \cos^2 x \text{ holds, is } \underline{\hspace{2cm}}$$

Official Ans. by NTA (4)

Allen Ans. (4)

$$\text{Sol. } x \in \left(\frac{\pi}{4}, \frac{7\pi}{4} \right)$$

$$14 \operatorname{cosec}^2 x - 2 \sin^2 x = 21 - 4 \cos^2 x$$

$$= 21 - 4(1 - \sin^2 x)$$

$$= 17 + 4 \sin^2 x$$

$$14 \operatorname{cosec}^2 x - 6 \sin^2 x = 17$$

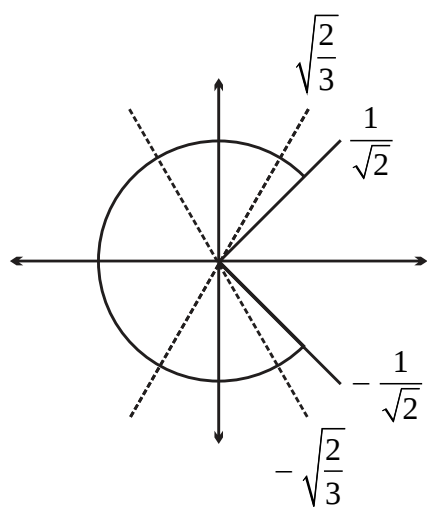
$$\text{let } \sin^2 x = p$$

$$\frac{14}{p} - 6p = 17 \Rightarrow 14 - 6p^2 = 17p$$

$$6p^2 + 17p - 14 = 0$$

$$p = -3.5, \frac{2}{3} \Rightarrow \sin^2 x = \frac{2}{3}$$

$$\Rightarrow \sin x = \pm \sqrt{\frac{2}{3}}$$



$\therefore$ Total 4 solutions

6. For a natural number n , let $a_n = 19^n - 12^n$. Then,

the value of $\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8}$ is

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $a_n = 19^n - 12^n$

$$\begin{aligned} \frac{31\alpha_9 - \alpha_{10}}{57\alpha_8} &= \frac{31(19^9 - 12^9) - (19^{10} - 12^{10})}{57\alpha_8} \\ &= \frac{19^9(31 - 19) - 12^9(31 - 12)}{57\alpha_8} \\ &= \frac{19^9 \cdot 12 - 12^{10} \cdot 19}{57\alpha_8} \\ &= \frac{12 \cdot 19(19^8 - 12^8)}{57\alpha_8} = 4 \end{aligned}$$

7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = \left(2 \left(1 - \frac{x^{25}}{2} \right) (2 + x^{25}) \right)^{\frac{1}{50}}. \text{ If the function}$$

$g(x) = f(f(f(x))) + f(f(x))$, the the greatest integer less than or equal to $g(1)$ is _____

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $f(x) = \left[2 \left(1 - \frac{x^{25}}{2} \right) (2 + x^{25}) \right]^{\frac{1}{50}}$

$$f(x) = \left[(2 - x^{25})(2 + x^{25}) \right]^{\frac{1}{50}} = (4 - x^{50})^{1/50}$$

$$f(f(x)) = \left(4 - \left((4 - x^{50})^{1/50} \right)^{50} \right)^{1/50} = x$$

$$g(x) = f(f(f(x))) + f(f(x)) = f(x) + x$$

$$g(1) = f(1) + 1 = 3^{1/50} + 1$$

$$[g(1)] = [3^{1/50} + 1] = 2$$

8. Let the lines

$$L_1 : \vec{r} = \lambda(\hat{i} + 2\hat{j} + 3\hat{k}), \lambda \in \mathbb{R}$$

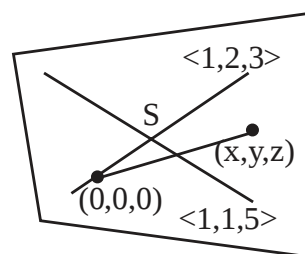
$$L_2 : \vec{r} = (\hat{i} + 3\hat{j} + \hat{k}) + \mu(\hat{i} + \hat{j} + 5\hat{k}); \mu \in \mathbb{R}$$

intersect at the point S . If a plane $ax + by - z + d = 0$ passes through S and is parallel to both the lines L_1 and L_2 , then the value of $a + b + d$ is equal to _____

Official Ans. by NTA (5)

Allen Ans. (5)

Sol. Both the lines lie in the same plane



$\therefore$ equation of the plane

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 1 & 1 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 7x - 2y - z = 0$$

$$\therefore a + b + d = 5$$

9. Let A be a 3×3 matrix having entries from the set $\{-1, 0, 1\}$. The number of all such matrices A having sum of all the entries equal to 5, is _____

Official Ans. by NTA (414)

Allen Ans. (414)

Sol. Case-I: $1 \rightarrow 7$ times
and $-1 \rightarrow 2$ times

$$\text{number of possible matrix} = \frac{9!}{7!2!} = 36$$

Case-II: $1 \rightarrow 6$ times,
 $-1 \rightarrow 1$ times
and $0 \rightarrow 2$ times

$$\text{number of possible matrix} = \frac{9!}{6!2!} = 252$$

Case-III: $1 \rightarrow 5$ times,
and $0 \rightarrow 4$ times

$$\text{number of possible matrix} = \frac{9!}{5!4!} = 126$$

Hence total number of all such matrix A
= 414

10. The greatest integer less than or equal to the sum of first 100 terms of the sequence

$$\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \dots \text{ is equal to}$$

Official Ans. by NTA (98)

Allen Ans. (98)

Sol. $\frac{1}{3} + \frac{5}{9} + \frac{19}{27} + \frac{65}{81} + \dots$

$$\left(1 - \frac{2}{3}\right) + \left(1 - \frac{4}{9}\right) + \left(1 - \frac{8}{27}\right) + \left(1 - \frac{16}{81}\right) \dots 100 \text{ terms}$$

$$100 - \left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots \right]$$

$$100 - \frac{\frac{2}{3} \left(1 - \left(\frac{2}{3}\right)^{100} \right)}{1 - \frac{2}{3}}$$

$$100 - 2 \left(1 - \left(\frac{2}{3}\right)^{100} \right)$$

$$S = 98 + 2 \left(\frac{2}{3}\right)^{100}$$

$$\Rightarrow [S] = 98$$