FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Thursday 28th July, 2022) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. Let the solution curve of the differential equation $xdy = \left(\sqrt{x^2+y^2}+y\right)dx,\ x>0$, intersect the line $x = 1$ at $y = 0$ and the line $x = 2$ at $y = \alpha$. Then the value of α is :

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $-\frac{3}{2}$ (D) $\frac{5}{2}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $xdy = \left(\sqrt{x^2+y^2}+y\right)dx$

$xdy - ydx = \sqrt{x^2+y^2}\,dx$

$\frac{xdy-ydx}{x^2} = \sqrt{1+\frac{y^2}{x^2}}\cdot\frac{dx}{x}$

$\frac{d\left(y/x\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{dx}{x}$

$\ln\left(\frac{y}{x}+\sqrt{\left(\frac{y}{x}\right)^2+1}\right) = \ln x + R$

$\frac{y+\sqrt{y^2+x^2}}{x} = cx$

$y+\sqrt{y^2+x^2} = cx^2$

$x = 1, y = 0 \Rightarrow 0+1 = C \Rightarrow C = 1$

Curve is $y+\sqrt{x^2+y^2} = x^2$

$x = 2, y = \alpha$

$2+\sqrt{4+\alpha^2} = 4$

$4+\alpha^2 = 16+\alpha^2 = 8\alpha$

$\alpha = \frac{3}{2}$

2. Considering only the principal values of the inverse trigonometric functions, the domain of the function $f(x) = \cos^{-1}\left(\frac{x^2-4x+2}{x^2+3}\right)$ is :

(A) $\left(-\infty, \frac{1}{4}\right]$ (B) $\left[-\frac{1}{4}, \infty\right)$

(C) $\left(-\frac{1}{3}, \infty\right)$ (D) $\left(-\infty, \frac{1}{3}\right]$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.

$\left|\frac{x^2+4x+2}{x^2+3}\right| \le 1$

$\Leftrightarrow \left(x^2-4x+2\right)^2 \le \left(x^2+3\right)^2$

$\Leftrightarrow \left(x^2-4x+2\right)^2 - \left(x^2+3\right)^2 \le 0$

$\Leftrightarrow \left(2x^2-4x+5\right)\left(-4x-1\right) \le 0$

$\Leftrightarrow -4x-1 \le 0 \rightarrow x \ge -\frac{1}{4}$

3. Let the vectors $\vec{a} = (1+t)\hat{i}+(1-t)\hat{j}+\hat{k}$, $\vec{b} = (1-t)\hat{i}+(1+t)\hat{j}+2\hat{k}$ and $\vec{c} = t\hat{i}-t\hat{j}+\hat{k}$, $t \in R$ be such that for $\alpha, \beta, \gamma \in R$, $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c} = \vec{0}$ $\Rightarrow \alpha = \beta = \gamma = 0$. Then, the set of all values of t is :

(A) a non-empty finite set

(B) equal to N

(C) equal to R – {0}

(D) equal to R

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. By its given condition

: $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors

$\Rightarrow \left[\vec{a}\,\vec{b}\,\vec{c}\right] \neq 0$ ..(i)

Now, $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$$= \begin{vmatrix} 1+t & 1-t & 1 \\ 1-t & 1+t & 2 \\ t & -t & 1 \end{vmatrix}$$

$C_2 \to C_1 + C_2$

$$\begin{vmatrix} 1+t & 2 & 1 \\ 1-t & 2 & 2 \\ t & 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1+t & 1 & 1 \\ 1-t & 1 & 2 \\ t & 0 & 1 \end{vmatrix}$$

$$= 2\left[(1+t)-(1-t)+t\right]$$

$$= 2[3t] = 6t$$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] \neq 0 \Rightarrow t \neq 0$$

4. Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) = \cos^{-1}(2x)$ is equal to :

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\cos^{-1}x = 2\sin^{-1}x = \cos^{-1}2x$

$$\cos^{-1}x - 2\left(\frac{\pi}{2} - \cos^{-1}x\right) = \cos^{-1}2x$$

$$\cos^{-1}x - \pi + 2\cos^{-1}x = \cos^{-1}2x$$

$$3\cos^{2}x = \pi + \cos^{-1}2x \quad \textbf{...(1)}$$

$$\cos\left(3\cos^{-1}x\right) = \cos\left(\pi + \cos^{-1}2x\right)$$

$$4x^3 - 3x = -2x$$

$$4x^3 = x \Rightarrow x = 0, \pm\frac{1}{2}$$

All satisfy the original equation

$$\text{sum} = -\frac{1}{2} \text{ to } + \frac{1}{2} = 0$$

5. Let the operations $*, \odot \in \{\wedge, \vee\}$. If $(p * q) \odot (p \odot \sim q)$ is a tautology, then the ordered pair $(*, \odot)$ is :

(A) $(\vee, \wedge)$ (B) $(\vee, \vee)$ (C) $(\wedge, \wedge)$ (D) $(\wedge, \vee)$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Well check each option

For A $\pi = \text{v of } 0 = \Lambda$

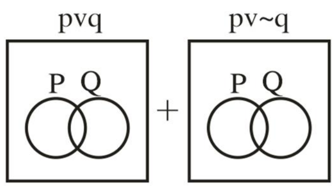
$$(pvq) \wedge (pv \sim q)$$

$$\equiv pv(q \wedge \sim q)$$

$$\equiv pv(c) \equiv p$$

For B : * = v, O = v

$(pvq) \vee (pv \sim q) \equiv t$ using Venn Diagrams



6. Let a vector $\vec{a}$ has a magnitude 9. Let a vector $\vec{b}$ be such that for every $(x, y) \in R \times R - \{(0,0)\}$, the vector $(x\vec{a} + y\vec{b})$ is perpendicular to the vector $(6y\vec{a} - 18x\vec{b})$. Then the value of $\left|\vec{a} \times \vec{b}\right|$ is equal to:

(A) $9\sqrt{3}$ (B) $27\sqrt{3}$ (C) 9 (D) 81

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $|\vec{a}| = 9$ & $\left(x\vec{a} + y\vec{b}\right) \cdot \left(6y\vec{a} - 18x\vec{b}\right) = 0$

$$\Rightarrow 6xy|\vec{a}|^2 - 18x^2\left(\vec{a}\cdot\vec{b}\right) + 6y^2\left(\vec{a}\cdot\vec{b}\right) - 18xy\left|\vec{b}\right|^2 = 0$$

$$\Rightarrow 6xy\left(|\vec{a}|^2 - 3\left|\vec{b}\right|^2\right) + \left(\vec{a}\cdot\vec{b}\right)\left(y^2 - 3x^2\right) = 0$$

This should hold $\forall\, x, y \in R \times R$

$$\therefore |\vec{a}|^2 = 3\left|\vec{b}\right|^2 \ \&\ \left(\vec{a}\cdot\vec{b}\right) = 0$$

Now $\left|\vec{a} \times \vec{b}\right|^2 = |\vec{a}|^2\left|\vec{b}\right|^2 - \left(\vec{a}\cdot\vec{b}\right)^2$

$$= |\vec{a}|^2 \cdot \frac{|\vec{a}|^2}{3}$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = \frac{|\vec{a}|^2}{\sqrt{3}} = \frac{81}{\sqrt{3}} = 27\sqrt{3}$$

7. For $t \in (0, 2\pi)$, if ABC is an equilateral triangle with vertices A(sint, –cost), B(cost, sint) and C(a, b) such that its orthocentre lies on a circle with centre $\left(1, \frac{1}{3}\right)$, then $(a^2 - b^2)$ is equal to :

(A) $\frac{8}{3}$ (B) 8

(C) $\frac{77}{9}$ (D) $\frac{80}{9}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $s \equiv \sin t, c \equiv \cos t$

Let orthocentre be (h,k)

Since it if an equilateral triangle hence orthocentre coincides with centroid.

$\therefore\ a+s+c=3h, b+s-c=3k$

$\therefore\ (3h-a)^2+(3k-b)^2=(s+c)^2+(s-c)^2=2(s^2+c^2)=2$

$\therefore\ \left(h-\frac{a}{3}\right)^2+\left(K-\frac{b}{3}\right)^2=\frac{2}{9},$

circle centre at $\left(\frac{a}{3},\frac{b}{3}\right)$

Gives, $\frac{a}{3}=1, \frac{b}{3}=\frac{1}{3} \Rightarrow a=3, b=1$

$\therefore\ a^2-b^2=8$

8. For $\alpha \in N$, consider a relation R on N given by $R=\{(x,y):3x+\alpha y \text{ is a multiple of } 7\}$. The relation R is an equivalence relation if and only if :

(A) $\alpha = 14$

(B) α is a multiple of 4

(C) 4 is the remainder when α is divided by 10

(D) 4 is the remainder when α is divided by 7

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. For R to be reflexive $\Rightarrow$ x R x

$\Rightarrow\ 3x+\alpha x=7x \Rightarrow (3+\alpha)x=7K$

$\Rightarrow\ 3+\alpha=7\lambda \Rightarrow \alpha=7\lambda-3=7N+4, K,\lambda,N \in I$

$\therefore$ when α divided by 7, remainder is 4.

R to be symmetric $xRy \Rightarrow yRx$

$3x+\alpha y=7N_1, 3y+\alpha x=7N_2$

$\Rightarrow (3+\alpha)(x+y)=7(N_1+N_2)=7N_3$

Which holds when $3+\alpha$ is multiple of 7

$\therefore\ \alpha=7N+4$ (as did earlier)

R to be transitive

$xRy\ \&\ yRz \Rightarrow xRz$.

$3x+\alpha y=7N_1\ \&\ 3y+\alpha z=7N_2$ and

$3x+\alpha z=7N_3$

$\therefore\ 3x+7N_2-3y=7N_3$

$\therefore\ 7N_1-\alpha y+7N_2-3y=7N_3$

$\therefore\ 7(N_1+N_2)-(3+\alpha)y=7N_3$

$\therefore\ (3+\alpha)y=7N$

Which is true again when $3+\alpha$ divisible by 7, i.e. when α divided by 7, remainder is 4.

9. Out of 60% female and 40% male candidates appearing in an exam, 60% candidates qualify it. The number of females qualifying the exam is twice the number of males qualifying it. A candidate is randomly chosen from the qualified candidates. The probability, that the chosen candidate is a female, is :

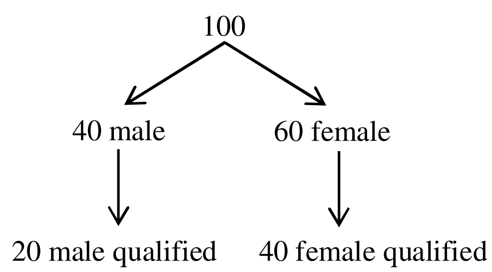
(A) $\frac{3}{4}$ (B) $\frac{11}{16}$

(C) $\frac{23}{32}$ (D) $\frac{13}{16}$

Official Ans. by NTA (A)

Allen Ans. (Bonus)

Sol.



Probability that chosen candidate is female = $\frac{40}{60}=\frac{2}{3}$

10. If $y = y(x),\ x \in \left(0, \frac{\pi}{2}\right)$ be the solution curve of the differential equation

$$\left(\sin^2 2x\right)\frac{dy}{dx} + \left(8\sin^2 2x + 2\sin 4x\right)y =$$

$$2e^{-4x}\left(2\sin 2x + \cos 2x\right), \quad \text{with} \quad y\left(\frac{\pi}{4}\right) = e^{-\pi},$$

then $y\left(\frac{\pi}{6}\right)$ is equal to :

(A) $\frac{2}{\sqrt{3}}e^{-2\pi/3}$ (B) $\frac{2}{\sqrt{3}}e^{2\pi/3}$

(C) $\frac{1}{\sqrt{3}}e^{-2\pi/3}$ (D) $\frac{1}{\sqrt{3}}e^{2\pi/3}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Given differential equation can be re-written as

$$\frac{dy}{dx} + \left(8 + 4\cot 2x\right)y = \frac{2e^{-4x}}{\sin^2 2x}\left(2\sin x + \cos 2x\right)$$

which is a linear diff. equation.

$$\text{I.f.} = e^{\int (8+4\cot 2x)dx} = e^{8x + 2Cu(\sin 2x)}$$

$$= e^{8x} \cdot \sin^2 2x$$

$\therefore$ solution is

$$y\left(e^{8x} \cdot \sin^2 2x\right) = \int 2e^{4x}\left(2\sin 2x + \cos 2x\right)dx + C$$

$$= e^{4x} \cdot \sin 2x + C$$

Given $y\left(\frac{\pi}{4}\right) = e^{-\pi} \Rightarrow C = 0$

$$\therefore\ y = \frac{e^{-4x}}{\sin 2x}$$

$$\therefore\ y\left(\frac{\pi}{6}\right) = \frac{e^{-4\cdot\frac{\pi}{6}}}{\sin\left(2\cdot\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}}e^{-\frac{2\pi}{3}}$$

11. If the tangents drawn at the points P and Q on the parabola $y^2 = 2x - 3$ intersect at the point R(0, 1), then the orthocentre of the triangle PQR is :

(A) (0, 1) (B) (2, –1)

(C) (6, 3) (D) (2, 1)

Official Ans. by NTA (B)

Allen Ans. (B)

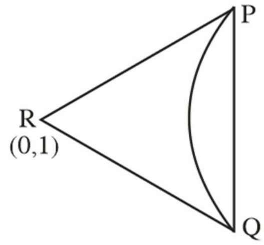
Sol. $y^2 = 2x - 3$ **...(i)**

Equation of chord of contact

PQ : $r = 0$

$yx1 = (x + 0) - 3$

$y = x - 3$ ...(2)



from (1) and (2)

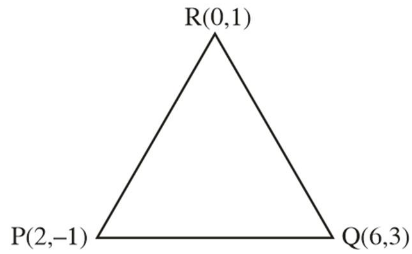
$$(x \cdot 3)^2 = 2x - 3$$

$$x^2 - 8x + 12 = 0$$

$$(x-2)(x-6) = 0$$

$$x = 2 \text{ or } 6$$

$$y = -1 \text{ or } 3$$



$$MPQ = \frac{1}{4} = 1$$

$$MQR = \frac{2}{6} = \frac{1}{3}$$

$$MPR = \frac{2}{6} = \frac{1}{3}$$

$$MPR = \frac{2}{-2} = -1$$

$$MPQ \times MPR = - \Rightarrow PQ \perp PR$$

Orthocentre = P (2,–1)

12. Let C be the centre of the circle $x^2 + y^2 - x + 2y = \frac{11}{4}$ and P be a point on the circle. A line passes through the point C, makes an angle of $\frac{\pi}{4}$ with the line CP and intersects the circle at the points Q and R. Then the area of the triangle PQR (in unit^2) is :

(A) 2 (B) $2\sqrt{2}$

(C) $8\sin\left(\frac{\pi}{8}\right)$ (D) $8\cos\left(\frac{\pi}{8}\right)$

Official Ans. by NTA (B)

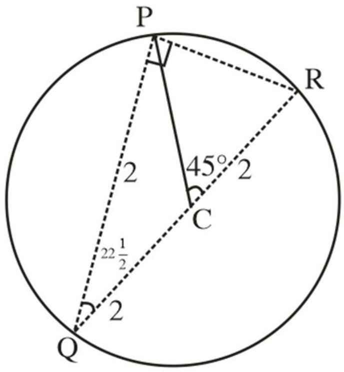
Allen Ans. (B)

Sol. $x^2 + y^2 - x + 2y = \frac{11}{4}$

$\left(x - \frac{1}{2}\right)^2 + (y+1)^2 = (2)^2$

Or Δ PQR

$PR = QK\sin 2 \geq \frac{1}{3}$



$= 4 \cdot 6\sin\frac{\pi}{8}$

$PQ = QR\cos 22\frac{1}{2}$

$= 4\cos\frac{\pi}{8}$

As $\Delta PQR = \frac{1}{2} PR \times PQ$

$= \frac{1}{2}\left(4^2 \sin\frac{\pi}{6}\right)\left(4\cos\frac{\pi}{8}\right)$

$= 4\sin\frac{\pi}{4} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

13. The remainder when $7^{2022} + 3^{2022}$ is divided by 5 is:

(A) 0 (B) 2 (C) 3 (D) 4

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $7^{2022} + 3^{2022}$

$= (49)^{1011} + (9)^{1011}$

$= (50-1)^{1011} + (10-1)^{1011}$

$= 5\lambda - 1 + 5K - 1$

$= 5m - 2$

Remainder $= 5 - 2 = 3$

14. Let the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and the matrix $B_0 = A^{49} + 2A^{98}$. If $B_n = \text{Adj}(B_{n-1})$ for all $n \geq 1$, then $\det(B_4)$ is equal to :

(A) 3^{28} (B) 3^{30} (C) 3^{32} (D) 3^{36}

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$a \leftrightarrow R_2$

$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$B_0 = A^{49} + 2A^{98}$

$= A + 2I$

$B_n = \text{Adj}(B_n - 1)$

$B_4 = \text{Adj}(\text{Adj}(\text{Adj}(\text{Adj}B_0))$

$= |B_0|^{(n-1)^4}$

$= |B_0|^{16}$

$$B_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

$= 2(4-0) - 1(0-1)$

$= 9$

$B_4(9)^{16} = (3)^{32}$

15. Let $S_1 = \left\{z_1 \in C : |z_1 - 3| = \frac{1}{2}\right\}$ and $S_2 = \{z_2 \in C : |z_2 - |z_2 + 1|| = |z_2 + |z_2 - 1||\}$. Then, for $z_1 \in S_1$ and $z_2 \in S_2$, the least value of $|z_2 - z_1|$ is :

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $|z_2 + |z_2 - 1||^2 = |z_2 - |z_2 + 1||^2$

$\Rightarrow |z_2 + |z_2 - 1||(\bar{z}_2 + |z_2 - 1|) = (z_2 - |z_2 + 1|)(\bar{z}_2 - (z_2 + 1))$

$\Rightarrow z_2|\bar{z}_2 + 12_2 - 1| - (\bar{z}_2 - |z_2 + 1|) + \bar{z}_2(|z_2 - 1| + |z_2 + 1|)$

$= |z_2 + 1|^2 = |z_2 - 1|^2$

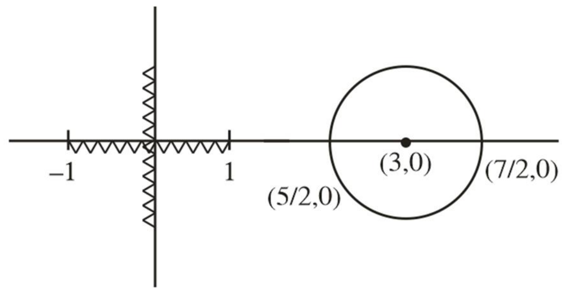
$\Rightarrow [z_2 + \bar{z}_2)(|z_2 - 1|) + (z_2 + 1|) = 2(z_2 + \bar{z}_2)$

$\Rightarrow (z_2 + \bar{z}_2)(|z_2 - 1| + |z_2 + 1| - 2) = 0$

$\therefore z_2 + \bar{z}_2 = 0$ or $|z_2 - 1| + |z_2 + 1| - 2 = 0$

$\therefore z_2$ lie on imaginary axis. Or on real axis with in [–1,1]

Also $|z_1 - 3| = \frac{1}{2}$ lie on circle having centre 3 and radius $\frac{1}{2}$.



Clearly $|z_1 - z_2|\text{min} = \frac{5}{2} - 1 = \frac{3}{2}$

16. The foot of the perpendicular from a point on the circle $x^2 + y^2 = 1$, $z = 0$ to the plane $2x + 3y + z = 6$ lies on which one of the following curves ?

(A) $(6x + 5y - 12)^2 + 4(3x + 7y - 8)^2 = 1$, $z = 6 - 2x - 3y$

(B) $(5x + 6y - 12)^2 + 4(3x + 5y - 9)^2 = 1$, $z = 6 - 2x - 3y$

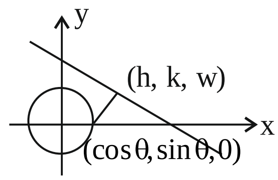
(C) $(6x + 5y - 14)^2 + 9(3x + 5y - 7)^2 = 1$, $z = 6 - 2x - 3y$

(D) $(5x + 6y - 14)^2 + 9(3x + 7y - 8)^2 = 1$, $z = 6 - 2x - 3y$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



$\frac{h - \cos\theta}{2} = \frac{k - \sin\theta}{3} = \frac{w - 0}{1}$

$= \frac{-1(2\cos\theta + 3\sin\theta - 6)}{14}$

$h = \cos\frac{-2(2\cos\theta + 3\sin\theta - 6)}{14}$

$= \frac{10\cos\theta - 6\sin\theta + 12}{14}$

$k = \sin\theta - \frac{3}{14}(2\cos\theta + 3\sin\theta - 6)$

$k = \frac{5\sin\theta - 6\cos\theta + 18}{14}$

Elementary $\sin\theta$ and $\cos\theta$

$(5h + 6k - 12)^2 + 4(3h + 5k - 9)^2 = 1$

17. If the minimum value of $f(x)=\frac{5x^2}{2}+\frac{\alpha}{x^5}, x>0$, is 14, then the value of α is equal to :

(A) 32 (B) 64

(C) 128 (D) 256

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\frac{x^2}{2}+\frac{x^2}{2}+\frac{x^2}{2}+\frac{x^2}{2}+\frac{x^2}{2}+\frac{\alpha}{2x^5}+\frac{\alpha}{2x^5}$

$\geq 7\left(\frac{\alpha^2}{2^7}\right)^{\frac{1}{7}}$

$\frac{7\cdot(\alpha)^{2/7}}{2}=14$

$(\alpha^2)^{1/7}=2^2$

$\alpha=\left(2^2\right)^{7/2}=2^7$

$\alpha=128$

18. Let α, β and γ be three positive real numbers. Let $f(x)=\alpha x^5+\beta x^3+\gamma x, x\in R$ and $g:R\to R$ be such that $g(f(x)) = x$ for all $x\in R$. If $a_1, a_2, a_3,..., a_n$ be in arithmetic progression with mean zero, then the value of $f\left(g\left(\frac{1}{n}\sum_{i=1}^{n}f(a_i)\right)\right)$ is equal to :

(A) 0 (B) 3

(C) 9 (D) 27

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Consider a case when $\alpha=\beta=0$ then

$f(x) = \gamma x$

$g(x)=\frac{x}{\gamma}$

$\frac{1}{n}\sum_{i=1}^{n}f(a_i)\Rightarrow\frac{\gamma}{n}(a_1+a_2+.....+a_n)$

$=0$

$\Rightarrow f(g(0))\Rightarrow f(0)$

$\Rightarrow 0$

19. Consider the sequence $a_1, a_2, a_3,$ …… such that $a_1 = 1, a_2 = 2$ and $a_{n+2}=\frac{2}{a_{n+1}}+a_n$ for n = 1, 2, 3, … .

If $\left(\frac{a_1+\frac{1}{a_2}}{a_3}\right)\cdot\left(\frac{a_2+\frac{1}{a_3}}{a_4}\right)\cdot\left(\frac{a_3+\frac{1}{a_4}}{a_5}\right).....\left(\frac{a_{30}+\frac{1}{a_{31}}}{a_{32}}\right)=2^\alpha\left({}^{61}C_{31}\right)$,

then α is equal to :

(A) –30 (B) –31

(C) –60 (D) –61

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $a_{n+2}\,a_{n+1}-a_{n+1}.a_n=2$

Series will satisfy

$a_1a_2, a_2a_3, a_3a_4, a_4a_5,$

1.2 2.2 2.3 2.4

$\frac{a_n+\frac{1}{a_{n+1}}}{a_{n+2}}=\frac{a_{n+2}-\frac{1}{a_{n+1}}}{a_{n+2}}$

$=1-\frac{1}{a_{n+1}a_{n+2}}$

$=1-\frac{1}{2(r+1)}$

$=\frac{2r+1}{2(r+1)}$

Now proof is given by

$=\prod_{r=1}^{30}\frac{(2r+1)}{2(r+1)}$

$=\frac{(1\cdot3\cdot5\cdot........\cdot61)}{2^{30}\cdot(2\cdot3\cdot.......\cdot31)}$

$\Rightarrow\frac{(1\cdot3\cdot5\cdot........\cdot61)}{\lfloor31\cdot2^{30}}\times\frac{2^{30}\times\lfloor30}{2^{30}\times\lfloor30}$

$=\frac{\lfloor61}{2^{60}\lfloor31\cdot\lfloor30}$

$\alpha=-60$

20. The minimum value of the twice differentiable function $f(x)=\int_0^x e^{x-t}f'(t)dt-(x^2-x+1)e^x,\ x\in R$, is :

(A) $-\frac{2}{\sqrt{e}}$ (B) $-2\sqrt{e}$

(C) $-\sqrt{e}$ (D) $\frac{2}{\sqrt{e}}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $f(x)=e^x\cdot\int_0^x \frac{f'(t)}{e^t}dt$

$$f'(x)=e^x\cdot\int_0^x \frac{f'(t)}{e^t}dt+e^x\cdot\frac{f'(x)}{e^x}$$

$$-\left[(2x-1)\cdot e^x+(x^2-x+1)\cdot e^x\right]$$

$$\int_0^x \frac{f'(t)}{e^t}dt=x^2+x$$

$$\frac{f'(x)}{e^x}=2x+1$$

$$f'(x)=(2x+1)\cdot e^x$$

$$f'(x)=0\Rightarrow x=-\frac{1}{2}$$

$$f(x)=(2x+1)\cdot e^x-2e^x+C$$

$$f(0)=-1$$

$-1 = 1-2+C$

$C=0$

$$f(x)=e^x(2x-1)$$

$$f\left(-\frac{1}{2}\right)=\frac{-2}{\sqrt{e}}$$

SECTION-B

1. Let S be the set of all passwords which are six to eight characters long, where each character is either an alphabet from {A, B, C, D, E} or a number from {1, 2, 3, 4, 5} with the repetition of characters allowed. If the number of passwords in S whose at least one character is a number from {1, 2, 3, 4, 5} is $\alpha\times 5^6$, then α is equal to _____.

Official Ans. by NTA (7073)

Allen Ans. (7073)

Sol. Required no. = Total – no character from {1, 2, 3, 4, 5}

$=(10^6-5^6)+(10^7-5^7)+(10^8-5^8)$

$=10^6(1+10+100)-5^6(1+5+25)$

$=10^6\times 111-5^6\times 31$

$=2^6\times 5^6\times 111-5^6\times 31$

$=5^6(2^6\times 111-31)$

$=5^6\times\underbrace{7073}_{\alpha}$

$\therefore \alpha=7073$

2. Let P(–2, –1, 1) and $Q\left(\frac{56}{17},\frac{43}{17},\frac{111}{17}\right)$ be the vertices of the rhombus PRQS. If the direction ratios of the diagonal RS are $\alpha,-1,\beta,$ where both α and β are integers of minimum absolute values, then $\alpha^2+\beta^2$ is equal to _________.

Official Ans. by NTA (450)

Allen Ans. (450)

Sol. $RS\equiv(\alpha,-1,\beta)$

DR of PQ $\equiv\left(\frac{56}{17}+2,\frac{43}{17}+1,\frac{111}{17}-1\right)$

$$\equiv\left(\frac{90}{17},\frac{60}{17},\frac{94}{17}\right)$$

$$\frac{90}{17}\alpha+\frac{60}{17}(-1)+\frac{94}{17}\beta=0$$

$$90\alpha+94\beta=60$$

$$\beta=\frac{60-90\alpha}{94}$$

$$\beta=\frac{30(2-3\alpha)}{94}$$

$$\beta=-30\frac{(3\alpha-2)}{94}$$

$$\beta=\frac{-15}{47}(3\alpha-2)$$

$$\Rightarrow\frac{\beta}{-15}=\frac{3\alpha-2}{47}$$

$$\Rightarrow\beta=-15,\ \alpha=-15$$

$$\alpha^2+\beta^2=225+225$$

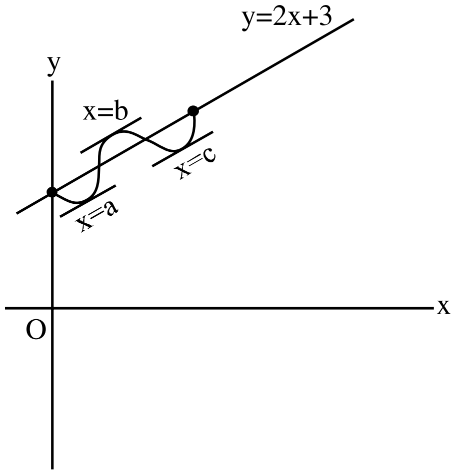
$=450$

3. Let $f:[0,1]\to R$ be a twice differentiable function in (0, 1) such that f(0) = 3 and f(1) = 5. If the line y = 2x + 3 intersects the graph of f at only two distinct points in (0, 1), then the least number of points $x\in(0,1)$, at which $f''(x)=0$, is ______.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.



$f'(a)=f'(b)=f'(c)=2$

$\Rightarrow f''(x)$ is zero

for atleast $x_1\in(a,b)$ & $x_2\in(b,c)$

4. If $\int_0^{\sqrt{3}}\frac{15x^3}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}dx=\alpha\sqrt{2}+\beta\sqrt{3}$, where α,β are integers, then $\alpha+\beta$ is equal to

Official Ans. by NTA (10)

Allen Ans. (10)

Sol. Put $1+x^2=t^2$

$2x\,dx=2t\,dt$

$X\,dx=t\,dt$

$$\therefore\int_1^2\frac{15(t^2-1)t\,dt}{\sqrt{t^2+t^3}}$$

$$15\int_1^2\frac{t(t^2-1)}{t\sqrt{1+t}}dt$$

Put $1+t=u^2$

$dt=2u\,du$

$$15\int_{\sqrt{2}}^{\sqrt{3}}\frac{(u^2-1)^2-1}{u}\times 2u\,du$$

$$30\int_{\sqrt{2}}^{\sqrt{3}}(u^4-2u^2)du$$

$$30\left(\frac{u^5}{5}-\frac{2u^3}{3}\right)_{\sqrt{2}}^{\sqrt{3}}$$

$$30\left[\frac{1}{5}\left(\sqrt{3}^5-\sqrt{2}^5\right)-\frac{2}{3}\left(\sqrt{3}^3-\sqrt{2}^3\right)\right]$$

$$30\left[\frac{1}{5}\left(9\sqrt{3}-4\sqrt{2}\right)-\frac{2}{3}\left(3\sqrt{3}-2\sqrt{2}\right)\right]$$

$$30\left[-\frac{1}{5}\times\sqrt{3}+\frac{8}{15}\sqrt{2}\right]$$

$$-6\sqrt{3}+16\sqrt{2}=\alpha\sqrt{2}+\beta\sqrt{3}$$

$\alpha=16,\ \beta=-6$

$\therefore\alpha+\beta=10$

5. Let $A=\begin{bmatrix}1&-1\\2&\alpha\end{bmatrix}$ and $B=\begin{bmatrix}\beta&1\\1&0\end{bmatrix}$, $\alpha,\beta\in R$. Let α_1 be the value of α which satisfies $(A+B)^2=A^2+\begin{bmatrix}2&2\\2&2\end{bmatrix}$ and α_2 be the value of α which satisfies $(A+B)^2=B^2$. Then $|\alpha_1-\alpha_2|$ is equal to ______.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $A+B=\begin{bmatrix}\beta+1&0\\3&\alpha\end{bmatrix}$

$$(A+B)^2=\begin{bmatrix}\beta+1&0\\3&\alpha\end{bmatrix}\begin{bmatrix}\beta+1&0\\3&\alpha\end{bmatrix}$$

$$=\begin{bmatrix}(\beta+1)^2&0\\3(\beta+1)+3\alpha&\alpha^2\end{bmatrix}$$

$$A^2=\begin{bmatrix}1&-1\\2&\alpha\end{bmatrix}\begin{bmatrix}1&-1\\2&\alpha\end{bmatrix}$$

$$=\begin{bmatrix}-1&-1-\alpha\\2+2\alpha&\alpha^2-2\end{bmatrix}$$

$$\therefore\begin{bmatrix}1&-\alpha+1\\2\alpha+4&\alpha^2\end{bmatrix}=\begin{bmatrix}(\beta+1)^2&0\\3(\alpha+\beta+1)&\alpha^2\end{bmatrix}$$

$\boxed{\alpha=1}=\alpha_1$

$$B^2=\begin{bmatrix}\beta&1\\1&0\end{bmatrix}\begin{bmatrix}\beta&1\\1&0\end{bmatrix}$$

$$=\begin{bmatrix}\beta^2+1&\beta\\\beta&1\end{bmatrix}=\begin{bmatrix}(\beta+1)^2&0\\3(\beta+1)+3\alpha&\alpha^2\end{bmatrix}$$

$\therefore\beta=0,\ \alpha=-1=\alpha_2$

$|\alpha_1-\alpha_2|=|1-(-1)|=2$

6. For $p, q \in R$, consider the real valued function $f(x) = (x-p)^2 - q$, $x \in R$ and $q > 0$. Let a_1, a_2, a_3 and a_4 be in an arithmetic progression with mean p and positive common difference. If $|f(a_i)| = 500$ for all i = 1, 2, 3, 4, then the absolute difference between the roots of f(x) = 0 is

Official Ans. by NTA (50)

Allen Ans. (50)

Sol. $f(x) = 0 \Rightarrow (x - p)^2 - q = 0$.

Roots are $p+\sqrt{q}$, $p-\sqrt{q}$ absolute difference between roots $2\sqrt{q}$.

Now, $|f(a_i)| = 500$

Let a_1, a_2, a_3, a_4 are a_1 a + d, a + 2d, a + 3d

$|f(a_4)| = 500$

$|(a_1 - p)^2 - q| = 500$

$\Rightarrow (a_1 - p)^2 - q = 500$

$\Rightarrow \frac{9}{4}d^2 - q = 500$ ________(1)

and $|f(a_1)|^2 = |f(a_2)|^2$

$((a_1 - p)^2 - q)^2 = ((a_2 - p)^2 - q)^2$

$\Rightarrow ((a_1 - p)^2 - (a_2 - p)^2)((a_1 - p)^2 - q + (a_2 - p)^2 - q) = 0$

$\Rightarrow \frac{9}{4}d^2 - q + \frac{d^2}{4} - q = 0$

$2q = \frac{10d^2}{4} \Rightarrow q = \frac{5d^2}{4}$

$\Rightarrow d^2 = \frac{4q}{5}$

From equation (1) $\frac{9}{4}.\frac{4.q}{5} - q = 500$

$\frac{4q}{5} = 500$

$\frac{4q}{5} = 500$

and $2\sqrt{q} = 2 \times \frac{50}{2} = 50$

7. For the hyperbola H : $x^2 - y^2 = 1$ and the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$, let the

(1) eccentricity of E be reciprocal of the eccentricity of H, and

(2) the line $y = \sqrt{\frac{5}{2}}x + K$ be a common tangent of E and H.

Then $4(a^2 + b^2)$ is equal to ________.

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $e_E = \sqrt{1 - \frac{b^2}{a^2}}$, $e_H = \sqrt{2}$

If $\Rightarrow e_E = \frac{1}{e_H}$

$\Rightarrow \frac{a^2 - b^2}{a^2} = \frac{1}{2}$

$2a^{2-2b}2 = a^2$

$a^2 = 2b^2$

and $y = \sqrt{\frac{5}{2}}x + k$ is tangent to ellipse then

$K^2 = a^2 \times \frac{5}{2} + b^2 = \frac{3}{2}$

$6b^2 = \frac{3}{2} \Rightarrow b^2 = \frac{1}{4}$ and $a^2 = \frac{1}{2}$

$\therefore 4.(a^2 + b^2) = 3$

8. Let $x_1, x_2, x_3, \ldots, x_{20}$ be in geometric progression with $x_1 = 3$ and the common ration $\frac{1}{2}$. A new data is constructed replacing each x_i by $(x_i - i)^2$. If $\overline{x}$ is the mean of new data, then the greatest integer less than or equal to $\overline{x}$ is __________.

Official Ans. by NTA (142)

Allen Ans. (142)

Sol. $$\sum x_0^1 = \frac{3\left(1-\left(\frac{1}{2}\right)\right)^{20}}{1\frac{-1}{2}} = 6\left(1-\frac{1}{2^{20}}\right)$$

$$=\sum_{i=1}^{20}\left(x_{i-i}\right)^2$$

$$=\sum_{i=1}^{20}\left(x_i\right)^2+\left(i\right)^2-2x_i i$$

Now $$=\sum_{i=1}^{20}\left(x_i\right)^2 = \frac{9\left(1-\left(\frac{1}{4}\right)\right)^{20}}{1-\frac{1}{4}} = 12\left(1-\frac{1}{2^{40}}\right)$$

$$\sum_{i=1}^{20} i^2 = \frac{1}{6}\times 20\times 21\times 41 = 2870$$

$$\sum_{i=1}^{20} x_i\ i = s = 3+2.3\frac{1}{2}+3.3\frac{1}{2^2}+4.3\frac{1}{2^3}+\text{.......AGP}$$

$$=6\left(2-\frac{22}{2^{20}}\right)$$

$$\bar{x} = \frac{12-\frac{12}{2^{40}}+2870-12\left(2-\frac{22}{2^{20}}\right)}{20}$$

$$\bar{x} = \frac{2858}{20}+\left(\frac{-12}{2^{40}}+\frac{22}{2^{20}}\right)\times\frac{1}{20}$$

$$\left[\bar{x}\right] = 142$$

9. $$\lim_{x\to 0}\left(\frac{\left(x+2\cos x\right)^3+2\left(x+2\cos x\right)^2+3\sin\left(x+2\cos x\right)}{\left(x+2\right)^3+2\left(x+2\right)^2+3\sin\left(x+2\right)}\right)^{\frac{100}{x}}$$

is equal to ____________.

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$$\lim_{x\to 10}\left(\frac{\left(x+2\cos x\right)^3+2\left(x+2\cos x\right)^2+3\sin\left(x+2\cos x\right)}{\left(x+2\right)^3+2\left(x+2\right)^2+3\sin\left(x+2\right)}\right)^{x}$$

Form 1^∞

$$= e^{\lim_{x\to 0}\left[\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)-1\right]\times\frac{100}{x}}$$

$$= e^{\lim_{x\to 0}\left[\frac{100}{x}\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)-\left((x+2)^3+2(x+2)^2+3\sin(x+2)\right)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)\right]}$$

$$= e^{\lim_{x\to 0}\frac{100}{x}\left[\left(\frac{(x+2\cos x)^3+(x+2)^3+2(x+2\cos x)^2-2(x+2)^2+3\sin(x+2\cos x)-3\sin(x+2)}{8+8+3\sin^2}\right)\right]}$$

$$= e^{\frac{100}{16+3\sin^2}\lim_{x\to 0}\frac{3(x+2\cos x)^2\times(1+2\sin x)-3(x+2)^2-4(x+2\cos x)}{x(1-2\sin x)-4(x+2)+3\cos(x+2\cos x)\times(1-2\sin x)-3\cos(x+2)}}$$

$$= e^{\frac{100}{16+3\sin 2}\left(\frac{12-3(4)+8\times 1-8+3\cos 2-3\cos 2}{1}\right)}$$

Using L'H rule.

$= e^o = 1$

10. The sum of all real values of x for which $\frac{3x^2-9x+17}{x^2+3x+10}=\frac{5x^2-7x+19}{3x^2+5x+12}$ is equal to ____.

Official Ans. by NTA (6)

Allen Ans. (6)

Sol. $$\frac{3x^2-9x+17}{x^2+3x+10}=\frac{5x^2-7x+19}{3x^2+5x+12}$$

$$\frac{x^2+3x+10+2x^2-12x+7}{x^2+3x+10}=\frac{3x^2+5x+12+2x^2-12x+7}{3x^2+5x+12}$$

$$1+\frac{2x^2-12x+7}{x^2+3x+10}=1+\frac{2x^2-12x+7}{3x^2+5x+12}$$

$$\left(2x^2-12x+7\right)\left(\frac{1}{x^2+3x+10}-\frac{1}{3x^2+5x+12}\right)=0$$

$2x^2-12x+7=0$ OR $3x^2+5x+12=x^2+3x+10$

$x=\frac{12\pm\sqrt{D}}{4}$ | $2x^2+2x+2=0$

$x^2+x+1=0$

Sum of Roots = 6 | No solution.