

FINAL JEE-MAIN EXAMINATION – JANUARY, 2023

(Held On Sunday 29th January, 2023)

TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

61. The statement $B \Rightarrow ((\sim A) \vee B)$ is equivalent to :

- (1) $B \Rightarrow (A \Rightarrow B)$
- (2) $A \Rightarrow (A \Leftrightarrow B)$
- (3) $A \Rightarrow ((\sim A) \Rightarrow B)$
- (4) $B \Rightarrow ((\sim A) \Rightarrow B)$

Official Ans. by NTA (1,3,4)

Allen Ans. (1 or 3 or 4)

Sol.

A	B	$\sim A$	$\sim A \vee B$	$B \Rightarrow ((\sim A) \vee B)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

$A \Rightarrow B$	$\sim A \Rightarrow B$	$B \Rightarrow (A \Rightarrow B)$	$A \Rightarrow ((\sim A) \Rightarrow B)$	$B \Rightarrow ((\sim A) \Rightarrow B)$
T	T	T	T	T
F	T	T	T	T
T	T	T	T	T
T	F	T	T	T

62. Shortest distance between the lines

$$\frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5} \text{ and } \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3} \text{ is}$$

- (1) $2\sqrt{3}$
- (2) $4\sqrt{3}$
- (3) $3\sqrt{3}$
- (4) $5\sqrt{3}$

Official Ans. by NTA (2)

Allen Ans. (2)

$$\text{Sol. } \frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5} \quad \vec{a} = \hat{i} - 8\hat{j} + 4\hat{k}$$

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3} \quad \vec{b} = \hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{p} = 2\hat{i} - 7\hat{j} + 5\hat{k}, \vec{q} = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\begin{aligned} \vec{p} \times \vec{q} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix} \\ &= \hat{i}(16) - \hat{j}(-16) + \hat{k}(16) \\ &= 16(\hat{i} + \hat{j} + \hat{k}) \end{aligned}$$

$$d = \left| \frac{(a-b) \cdot (\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|} \right| = \left| \frac{(-10\hat{j} - 2\hat{k}) \cdot 16(\hat{i} + \hat{j} + \hat{k})}{16\sqrt{3}} \right| \\ = \left| \frac{-12}{\sqrt{3}} \right| = 4\sqrt{3}$$

63. If $\vec{a} = \hat{i} + 2\hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = 7\hat{i} - 3\hat{k} + 4\hat{k}$, $\vec{r} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$ and $\vec{r} \cdot \vec{a} = 0$ then $\vec{r} \cdot \vec{c}$ is equal to :

- (1) 34
- (2) 12
- (3) 36
- (4) 30

Official Ans. by NTA (1)

Allen Ans. (1)

$$\vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$$

$$\Rightarrow (\vec{r} - \vec{c}) \times \vec{b} = \vec{0}$$

$$\Rightarrow \vec{r} - \vec{c} = \lambda \vec{b}$$

$$\Rightarrow \vec{r} = \vec{c} + \lambda \vec{b}$$

And given that $\vec{r} \cdot \vec{a} = 0$

$$\Rightarrow (\vec{c} + \lambda \vec{b}) \cdot \vec{a} = 0$$

$$\Rightarrow \vec{c} \cdot \vec{a} + \lambda \vec{b} \cdot \vec{a} = 0$$

$$\Rightarrow \lambda = \frac{-\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}}$$

$$\text{Now } \vec{r} \cdot \vec{c} = (\vec{c} + \lambda \vec{b}) \cdot \vec{c}$$

$$= \left(\vec{c} - \frac{\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}} \vec{b} \right) \cdot \vec{c}$$

$$= |\vec{c}| - \left(\frac{\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}} \right) (\vec{b} \cdot \vec{c})$$

$$= 74 - \left[\frac{15}{3} \right] 8$$

$$= 74 - 40 = 34$$

64. Let $S = \{w_1, w_2, \dots\}$ be the sample space associated to a random experiment. Let $P(w_n) = \frac{P(w_{n-1})}{2}, n \geq 2$.

Let $A = \{2k + 3\ell; k, \ell \in \mathbb{N}\}$ and $B = \{w_n; n \in A\}$. Then $P(B)$ is equal to

- (1) $\frac{3}{32}$ (2) $\frac{3}{64}$
(3) $\frac{1}{16}$ (4) $\frac{1}{32}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Let $P(w_1) = \lambda$ then $P(w_2) = \frac{\lambda}{2} \dots P(w_n) = \frac{\lambda}{2^{n-1}}$

$$\text{As } \sum_{k=1}^{\infty} P(w_k) = 1 \Rightarrow \frac{\lambda}{1 - \frac{1}{2}} = 1 \Rightarrow \lambda = \frac{1}{2}$$

$$\text{So, } P(w_n) = \frac{1}{2^n}$$

$$A = \{2k + 3\ell; k, \ell \in \mathbb{N}\} = \{5, 7, 8, 9, 10, \dots\}$$

$$B = \{w_n : n \in A\}$$

$$B = \{w_5, w_7, w_8, w_9, w_{10}, w_{11}, \dots\}$$

$$A = \mathbb{N} - \{1, 2, 3, 4, 6\}$$

$$\therefore P(B) = 1 - [P(w_1) + P(w_2) + P(w_3) + P(w_4) + P(w_6)]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{64} \right]$$

$$= 1 - \frac{32 + 16 + 8 + 4 + 1}{64} = \frac{3}{64}$$

65. The value of the integral $\int_1^2 \left(\frac{t^4 + 1}{t^6 + 1} \right) dt$ is :

- (1) $\tan^{-1} \frac{1}{2} + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}$
(2) $\tan^{-1} 2 - \frac{1}{3} \tan^{-1} 8 + \frac{\pi}{3}$
(3) $\tan^{-1} 2 + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}$
(4) $\tan^{-1} \frac{1}{2} - \frac{1}{3} \tan^{-1} 8 + \frac{\pi}{3}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $I = \int_1^2 \left(\frac{t^4 + 1}{t^6 + 1} \right) dt$

$$= \int_1^2 \frac{(t^4 + 1 - t^2) + t^2}{(t^2 + 1)(t^4 - t^2 + 1)} dt$$

$$= \int_1^2 \left(\frac{1}{t^2 + 1} + \frac{t^2}{t^6 + 1} \right) dt$$

$$= \int_1^2 \left(\frac{1}{t^2 + 1} + \frac{1}{3} \frac{3t^2}{(t^3)^2 + 1} \right) dt$$

$$= \tan^{-1}(t) + \frac{1}{3} \tan^{-1}(t^3) \Big|_1^2$$

$$= (\tan^{-1}(2) - \tan^{-1}(1)) + \frac{1}{3} (\tan^{-1}(2^3) - \tan^{-1}(1^3))$$

$$= \tan^{-1}(2) + \frac{1}{3} \tan^{-1}(8) - \frac{\pi}{3}$$

66. Let K be the sum of the coefficients of the odd powers of x in the expansion of $(1+x)^{99}$. Let a be

the middle term in the expansion of $\left(2 + \frac{1}{\sqrt{2}}\right)^{200}$. If

$$\frac{^{200}C_{99}K}{a} = \frac{2^\ell m}{n}, \text{ where } m \text{ and } n \text{ are odd numbers,}$$

then the ordered pair (ℓ, n) is equal to :

(1) (50, 51)

(2) (51, 99)

(3) (50, 101)

(4) (51, 101)

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. In the expansion of

$$(1+x)^{99} = C_0 + C_1 x + C_2 x^2 + \dots + C_{99} x^{99}$$

$$K = C_1 + C_3 + \dots + C_{99} = 2^{98}$$

a \Rightarrow Middle in the expansion of $\left(2 + \frac{1}{\sqrt{2}}\right)^{200}$

$$T_{\frac{200}{2}+1} = {}^{200}C_{100} (2)^{100} \left(\frac{1}{\sqrt{2}}\right)^{100}$$

$$= {}^{200}C_{100} \cdot 2^{50}$$

$$\text{So, } \frac{{}^{200}C_{99} \times 2^{98}}{{}^{200}C_{100} \times 2^{50}} = \frac{100}{101} \times 2^{48}$$

$$\text{So, } \frac{25}{101} \times 2^{50} = \frac{m}{n} 2^\ell$$

$\therefore m, n$ are odd so

(ℓ, n) become $(50, 101)$ Ans.

- 67.** Let f and g be twice differentiable functions on \mathbb{R} such that

$$f''(x) = g''(x) + 6x$$

$$f'(1) = 4g'(1) - 3 = 9$$

$$f(2) = 3g(2) = 12$$

Then which of the following is NOT true ?

(1) $g(-2) - f(-2) = 20$

(2) If $-1 < x < 2$, then $|f(x) - g(x)| < 8$

(3) $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1$

(4) There exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $f''(x) = g''(x) + 6x \quad \dots(1)$

$$f'(1) = 4g'(1) - 3 = 9 \quad \dots(2)$$

$$f(2) = 3g(2) = 12 \quad \dots(3)$$

By integrating (1)

$$f'(x) = g'(x) + 6 \frac{x^2}{2} + C$$

At $x = 1$,

$$f'(1) = g'(1) + 3 + C$$

$$\Rightarrow 9 = 4 + 3 + C \Rightarrow C = 3$$

$$\therefore f'(x) = g'(x) + 3x^2 + 3$$

Again by integrating,

$$f(x) = g(x) + \frac{3x^3}{3} + 3x + D$$

At $x = 2$,

$$f(2) = g(2) + 8 + 3(2) + D$$

$$\Rightarrow 12 = 4 + 8 + 6 + D \Rightarrow D = -6$$

$$\text{So, } f(x) = g(x) + x^3 + 3x - 6$$

$$\Rightarrow f(x) - g(x) = x^3 + 3x - 6$$

At $x = -2$,

$$\Rightarrow g(-2) - f(-2) = 20 \quad (\text{Option (1) is true})$$

Now, for $-1 < x < 2$

$$h(x) = f(x) - g(x) = x^3 + 3x - 6$$

$$\Rightarrow h'(x) = 3x^2 + 3$$

$$\Rightarrow h(x) \uparrow$$

$$\text{So, } h(-1) < h(x) < h(2)$$

$$\Rightarrow -10 < h(x) < 8$$

$$\Rightarrow |h(x)| < 10 \quad (\text{option (2) is NOT true})$$

$$\text{Now, } h'(x) = f'(x) - g'(x) = 3x^2 + 3$$

$$\text{If } |h'(x)| < 6 \Rightarrow |3x^2 + 3| < 6$$

$$\Rightarrow 3x^2 + 3 < 6$$

$$\Rightarrow x^2 < 1$$

$$\Rightarrow -1 < x < 1 \quad (\text{option (3) is True})$$

$$\text{If } x \in (-1, 1) \quad |f'(x) - g'(x)| < 6$$

option (3) is true and now to solve

$$f(x) - g(x) = 0$$

$$\Rightarrow x^3 + 3x - 6 = 0$$

$$h(x) = x^3 + 3x - 6$$

$$\text{here, } h(1) = -\text{ve} \text{ and } h\left(\frac{3}{2}\right) = +\text{ve}$$

$$\text{So there exists } x_0 \in \left(1, \frac{3}{2}\right) \text{ such that } f(x_0) = g(x_0)$$

(option (4) is true)

68. The set of all values of $t \in \mathbb{R}$, for which the matrix

$$\begin{bmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{bmatrix} \quad \text{is}$$

invertible, is

- (1) $\left\{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\right\}$ (2) $\left\{k\pi + \frac{\pi}{4}, k \in \mathbb{Z}\right\}$
(3) $\{k\pi, k \in \mathbb{Z}\}$ (4) \mathbb{R}

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. If its invertible, then determinant value $\neq 0$

So,

$$\begin{vmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{vmatrix} \neq 0$$

$$\Rightarrow e^t \cdot e^{-t} \cdot e^{-t} \begin{vmatrix} 1 & \sin t - 2\cos t & -2\sin t - \cos t \\ 1 & 2\sin t + \cos t & \sin t - 2\cos t \\ 1 & \cos t & \sin t \end{vmatrix} \neq 0$$

Applying, $R_1 \rightarrow R_1 - R_2$ then $R_2 \rightarrow R_2 - R_3$

We get

$$e^{-t} \begin{vmatrix} 0 & -\sin t - \cos t & -3\sin t + \cos t \\ 0 & 2\sin t & -2\cos t \\ 1 & \cos t & \sin t \end{vmatrix} \neq 0$$

By expanding we have,

$$e^{-t} \times 1 \left(2\sin t \cos t + 6\cos^2 t + 6\sin^2 t - 2\sin t \cos t \right) \neq 0$$

$$\Rightarrow e^{-t} \times 6 \neq 0$$

for $\forall t \in \mathbb{R}$

69. The area of the region

$$A = \left\{ (x, y) : |\cos x - \sin x| \leq y \leq \sin x, 0 \leq x \leq \frac{\pi}{2} \right\}$$

$$(1) 1 - \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{5}} \quad (2) \sqrt{5} + 2\sqrt{2} - 4.5$$

$$(3) \frac{3}{\sqrt{5}} - \frac{3}{\sqrt{2}} + 1 \quad (4) \sqrt{5} - 2\sqrt{2} + 1$$

Official Ans. by NTA (4)

Allen Ans. (4)

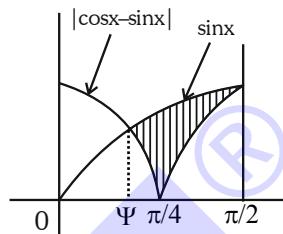
Sol. $|\cos x - \sin x| \leq y \leq \sin x$

Intersection point of $\cos x - \sin x = \sin x$

$$\Rightarrow \tan x = \frac{1}{2}$$

$$\text{Let } \psi = \tan^{-1} \frac{1}{2}$$

$$\text{So, } \tan \psi = \frac{1}{2}, \sin \psi = \frac{1}{\sqrt{5}}, \cos \psi = \frac{2}{\sqrt{5}}$$



$$\text{Area} = \int_{\psi}^{\pi/2} (\sin x - |\cos x - \sin x|) dx$$

$$= \int_{\psi}^{\pi/4} (\sin x - (\cos x - \sin x)) dx$$

$$+ \int_{\pi/4}^{\pi/2} (\sin x - (\sin x - \cos x)) dx$$

$$= \int_{\psi}^{\pi/4} (2\sin x - \cos x) dx + \int_{\pi/4}^{\pi/2} \cos x dx$$

$$= [-2\cos x - \sin x]_{\psi}^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}$$

$$= -\sqrt{2} - \frac{1}{\sqrt{2}} + 2\cos \psi + \sin \psi + \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= -\sqrt{2} - \frac{1}{\sqrt{2}} + 2\left(\frac{2}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}}\right) + 1 - \frac{1}{\sqrt{2}}$$

$$= \sqrt{5} - 2\sqrt{2} + 1$$

70. The set of all values of λ for which the equation $\cos^2 2x - 2\sin^4 x - 2\cos^2 x = \lambda$

$$(1) [-2, -1] \quad (2) \left[-2, -\frac{3}{2}\right]$$

$$(3) \left[-1, -\frac{1}{2}\right] \quad (4) \left[-\frac{3}{2}, -1\right]$$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\lambda = \cos^2 x - 2\sin^4 x - 2\cos^2 x$
convert all in to cos x.

$$\begin{aligned}\lambda &= (2\cos^2 x - 1)^2 - 2(1 - \cos^2 x)^2 - 2\cos^2 x \\&= 4\cos^4 x - 4\cos^2 x + 1 - 2(1 - 2\cos^2 x + \cos^4 x) - 2\cos^2 x \\&= 2\cos^4 x - 2\cos^2 x + 1 - 2 \\&= 2\cos^4 x - 2\cos^2 x - 1 \\&= 2\left[\cos^4 x - \cos^2 x - \frac{1}{2}\right] \\&= 2\left[\left(\cos^2 x - \frac{1}{2}\right)^2 - \frac{3}{4}\right] \\&\lambda_{\max} = 2\left[\frac{1}{4} - \frac{3}{4}\right] = 2 \times \left(-\frac{2}{4}\right) = -1 \text{ (max Value)} \\&\lambda_{\min} = 2\left[0 - \frac{3}{4}\right] = -\frac{3}{2} \text{ (MinimumValue)} \\&\text{So, Range} = \left[-\frac{3}{2}, -1\right]\end{aligned}$$

71. The letters of the word OUGHT are written in all possible ways and these words are arranged as in a dictionary, in a series. Then the serial number of the word TOUGH is :

- (1) 89 (2) 84
(3) 86 (4) 79

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Lets arrange the letters of OUGHT in alphabetical order.

G, H, O, T, U

Words starting with

G ----- $\rightarrow 4!$

H ----- $\rightarrow 4!$

O ----- $\rightarrow 4!$

T G --- $\rightarrow 3!$

T H --- $\rightarrow 3!$

T O G -- $\rightarrow 2!$

T O H -- $\rightarrow 2!$

T O U G H $\rightarrow 1!$

Total = 89

72. The plane $2x - y + z = 4$ intersects the line segment joining the points A(a, -2, 4) and B(2, b, -3) at the point C in the ratio 2 : 1 and the distance of the point C from the origin is $\sqrt{5}$. If $ab < 0$ and P is the point (a - b, b, 2b - a) then CP^2 is equal to :

- (1) $\frac{17}{3}$ (2) $\frac{16}{3}$
(3) $\frac{73}{3}$ (4) $\frac{97}{3}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. A(a, -2, 4), B(2, b, -3)

$$AC : CB = 2 : 1$$

$$\Rightarrow C \equiv \left(\frac{a+4}{3}, \frac{2b-2}{3}, \frac{-2}{3} \right)$$

$$C \text{ lies on } 2x - y + z = 4$$

$$\Rightarrow \frac{2a+8}{3} - \frac{2b-2}{3} - \frac{2}{3} = 4$$

$$\Rightarrow a - b = 2 \dots (1)$$

$$\text{Also } OC = \sqrt{5}$$

$$\Rightarrow \left(\frac{a+4}{3} \right)^2 + \left(\frac{2b-2}{3} \right)^2 + \left(\frac{-2}{3} \right)^2 = 5 \dots (2)$$

Solving, (1) and (2)

$$(b+6)^2 + (2b-2)^2 = 41$$

$$\Rightarrow 5b^2 + 4b - 1 = 0$$

$$\Rightarrow b = -1 \text{ or } \frac{1}{5}$$

$$\Rightarrow a = 1 \text{ or } \frac{11}{5}$$

But $ab < 0 \Rightarrow (a, b) = (1, -1)$

$$C \equiv \left(\frac{5}{3}, \frac{-4}{3}, \frac{-2}{3} \right), P \equiv (2, -1, -3)$$

$$CP^2 = \frac{1}{9} + \frac{1}{9} + \frac{49}{9} = \frac{51}{9} = \frac{17}{3}$$

73. Let $\vec{a} = 4\hat{i} + 3\hat{j}$ and $\vec{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ and \vec{c} is a vector such that $\vec{c} \cdot (\vec{a} \times \vec{b}) + 25 = 0$, $\vec{c} \cdot (\hat{i} + \hat{j} + \hat{k}) = 4$ and projection of \vec{c} on \vec{a} is 1, then the projection of \vec{c} on \vec{b} equals :

(1) $\frac{5}{\sqrt{2}}$

(2) $\frac{1}{5}$

(3) $\frac{1}{\sqrt{2}}$

(4) $\frac{3}{\sqrt{2}}$

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $\vec{a} \times \vec{b} = 15\hat{i} - 20\hat{j} - 25\hat{k}$

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow 15x - 20y - 25z + 25 = 0$$

$$\Rightarrow 3x - 4y - 5z = -5$$

Also $x + y + z = 4$

and $\frac{\vec{c} \cdot \vec{a}}{|\vec{a}|} = 1 \Rightarrow 4x + 3y = 5$

$$\Rightarrow \vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$$

Projection of \vec{c} or $\vec{b} = \frac{25}{5\sqrt{2}} = \frac{5}{\sqrt{2}}$

74. If the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+3}{1}$ and

$$\frac{x-a}{2} = \frac{y+2}{3} = \frac{z-3}{1}$$

intersect at the point P, then

- (1) 16 (2) 28
(3) 10 (4) 22

Official Ans. by NTA (2)
Allen Ans. (2)
Sol. Point on $L_1 \equiv (\lambda + 1, 2\lambda + 2, \lambda - 3)$

Point on $L_2 \equiv (2\mu + a, 3\mu - 2, \mu + 3)$

$$\lambda - 3 = \mu + 3 \Rightarrow \lambda = \mu + 6 \dots (1)$$

$$2\lambda + 2 = 3\mu - 2 \Rightarrow 2\lambda = 3\mu - 4 \dots (2)$$

Solving, (1) and (2)

$$\Rightarrow \lambda = 22 \text{ & } \mu = 16$$

$$\Rightarrow P \equiv (23, 46, 19)$$

$$\Rightarrow a = -9$$

Distance of P from $z = -9$ is 28

75. The value of the integral $\int_{1/2}^2 \frac{\tan^{-1} x}{x} dx$ is equal to

(1) $\pi \log_e 2$

(2) $\frac{1}{2} \log_e 2$

(3) $\frac{\pi}{4} \log_e 2$

(4) $\frac{\pi}{2} \log_e 2$

Official Ans. by NTA (4)
Allen Ans. (4)

Sol. $I = \int_{1/2}^2 \frac{\tan^{-1} x}{x} dx \dots \text{(i)}$

Put $x = \frac{1}{t}$ $dx = -\frac{1}{t^2} dt$

$$I = - \int_2^{1/2} \frac{\tan^{-1} \frac{1}{t}}{\frac{1}{t}} \cdot \frac{1}{t^2} dt = - \int_2^{1/2} \frac{\tan^{-1} \frac{1}{t}}{t} dt$$

$$I = \int_{1/2}^2 \frac{\cot^{-1} t}{t} dt = \int_{1/2}^2 \frac{\cot^{-1} x}{x} dx \dots \text{(ii)}$$

Add both equation

$$2I = \int_{1/2}^2 \frac{\tan^{-1} x + \cot^{-1} x}{x} dx = \frac{\pi}{2} \int_{1/2}^2 \frac{dx}{x} = \frac{\pi}{2} (\ell n 2)_{1/2}^2$$

$$= \frac{\pi}{2} \left(\ell n 2 - \ell n \frac{1}{2} \right) = \pi \ell n 2$$

$$I = \frac{\pi}{2} \ell n 2$$

76. If the tangent at a point P on the parabola $y^2 = 3x$ is parallel to the line $x + 2y = 1$ and the tangents at the points Q and R on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ are perpendicular to the line $x - y = 2$, then the area of the triangle PQR is:

(1) $\frac{9}{\sqrt{5}}$

(2) $5\sqrt{3}$

(3) $\frac{3}{2}\sqrt{5}$

(4) $3\sqrt{5}$

Official Ans. by NTA (4)
Allen Ans. (4)

Sol. $y^2 = 3x$

Tangent P(x_1, y_1) is parallel to $x + 2y = 1$

Then slope at P = $-\frac{1}{2}$

$$2y \frac{dy}{dx} = 3$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2y} = -\frac{1}{2}$$

$$\Rightarrow y_1 = -3$$

Coordinates of P(3, -3)

$$\text{Similarly } Q\left(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{5}}\right), R\left(-\frac{4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$$

Area of ΔPQR

$$= \frac{1}{2} \begin{vmatrix} 3 & -3 & 1 \\ \frac{4}{\sqrt{3}} & \frac{1}{\sqrt{5}} & 1 \\ -\frac{4}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[3\left(\frac{2}{\sqrt{5}}\right) + 3\left(\frac{8}{\sqrt{5}}\right) + 0 \right] = \frac{30}{2\sqrt{5}} = 3\sqrt{5}$$

77. Let $y = y(x)$ be the solution of the differential

equation $x \log_e x \frac{dy}{dx} + y = x^2 \log_e x, (x > 1)$. If

$y(2) = 2$, then $y(e)$ is equal to

$$(1) \frac{4+e^2}{4}$$

$$(2) \frac{1+e^2}{4}$$

$$(3) \frac{2+e^2}{2}$$

$$(4) \frac{1+e^2}{2}$$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $x \log_e x \frac{dy}{dx} + y = x^2 \log_e x, (x > 1)$.

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \ln x} = x$$

Linear differential equation

$$\text{I.F.} = e^{\int \frac{1}{x \ln x} dx} = |\ln x|$$

\therefore Solution of differential equation

$$y|\ln x| = \int x|\ln x| dx$$

$$= |\ln x| \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$\Rightarrow y|\ln x| = |\ln x| \left(\frac{x^2}{2} \right) - \frac{x^2}{4} + c$$

For constant

$$y(2) = 2 \Rightarrow c = 1$$

$$\text{So, } y(x) = \frac{x^2}{2} - \frac{x^2}{4|\ln x|} + \frac{1}{|\ln x|}$$

$$\text{Hence, } y(e) = \frac{e^2}{2} - \frac{e^2}{4} + 1 = 1 + \frac{e^2}{4}$$

78. The number of 3 digit numbers, that are divisible

by either 3 or 4 but not divisible by 48, is

$$(1) 472 \quad (2) 432$$

$$(3) 507 \quad (4) 400$$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Total 3 digit number = 900

$$\text{Divisible by 3} = 300 \quad (\text{Using } \frac{900}{3} = 300)$$

$$\text{Divisible by 4} = 225 \quad (\text{Using } \frac{900}{4} = 225)$$

$$\text{Divisible by 3 \& 4} = 108, \dots$$

$$(\text{Using } \frac{900}{12} = 75)$$

Number divisible by either 3 or 4

$$= 300 + 225 - 75 = 450$$

We have to remove divisible by 48,

$$144, 192, \dots, 18 \text{ terms}$$

$$\text{Required number of numbers} = 450 - 18 = 432$$

79. Let R be a relation defined on \mathbb{N} as a $R b$ is $2a + 3b$ is a multiple of 5, $a, b \in \mathbb{N}$. Then R is
- not reflexive
 - transitive but not symmetric
 - symmetric but not transitive
 - an equivalence relation

Official Ans. by NTA (4)
Allen Ans. (4)
Sol. $a R a \Rightarrow 5a$ is multiple of 5

So reflexive

$$a R b \Rightarrow 2a + 3b = 5\alpha,$$

Now $b R a$

$$\begin{aligned} 2b + 3a &= 2b + \left(\frac{5\alpha - 3b}{2} \right) \cdot 3 \\ &= \frac{15}{2}\alpha - \frac{5}{2}b = \frac{5}{2}(3\alpha - b) \\ &= \frac{5}{2}(2a + 2b - 2\alpha) \\ &= 5(a + b - \alpha) \end{aligned}$$

Hence symmetric

$$a R b \Rightarrow 2a + 3b = 5\alpha.$$

$$b R c \Rightarrow 2b + 3c = 5\beta$$

$$\text{Now } 2a + 5b + 3c = 5(\alpha + \beta)$$

$$\Rightarrow 2a + 5b + 3c = 5(\alpha + \beta)$$

$$\Rightarrow 2a + 3c = 5(\alpha + \beta - b)$$

$$\Rightarrow a R c$$

Hence relation is equivalence relation.

80. Consider a function $f: \mathbb{N} \rightarrow \mathbb{R}$, satisfying

$$f(1) + 2f(2) + 3f(3) + \dots + xf(x) = x(x+1)f(x); x \geq 2$$

with $f(1)=1$. Then $\frac{1}{f(2022)} + \frac{1}{f(2028)}$ is equal to

- 8200
- 8000
- 8400
- 8100

Official Ans. by NTA (4)
Allen Ans. (4)
Sol. Given for $x \geq 2$

$$f(1) + 2f(2) + \dots + xf(x) = x(x+1)f(x)$$

replace x by $x+1$

$$\Rightarrow x(x+1)f(x) + (x+1)f(x+1)$$

$$= (x+1)(x+2)f(x+1)$$

$$\Rightarrow \frac{x}{f(x+1)} + \frac{1}{f(x)} = \frac{(x+2)}{f(x)}$$

$$\Rightarrow x f(x) = (x+1) f(x+1) = \frac{1}{2}, x \geq 2$$

$$f(2) = \frac{1}{4}, f(3) = \frac{1}{6}$$

$$\text{Now } f(2022) = \frac{1}{4044}$$

$$f(2028) = \frac{1}{4056}$$

$$\text{So, } \frac{1}{f(2022)} + \frac{1}{f(2028)} = 4044 + 4056 = 8100$$

SECTION-B

81. The total number of 4-digit numbers whose greatest common divisor with 54 is 2, is _____.

Official Ans. by NTA (3000)
Allen Ans. (3000)
Sol. N should be divisible by 2 but not by 3

 $N = (\text{Numbers divisible by 2}) - (\text{Numbers divisible by 6})$

$$N = \frac{9000}{2} - \frac{9000}{6} = 4500 - 1500 = 3000$$

82. A triangle is formed by the tangents at the point $(2, 2)$ on the curves $y^2 = 2x$ and $x^2 + y^2 = 4x$, and the line $x + y + 2 = 0$. If r is the radius of its circumcircle, then r^2 is equal to _____.

Official Ans. by NTA (10)
Allen Ans. (10)

Sol. $S_1 : y^2 = 2x$ $S_2 : x^2 + y^2 = 4x$

$P(2,2)$ is common point on S_1 & S_2

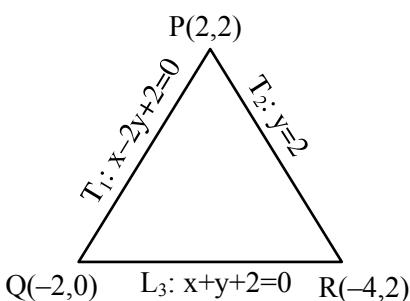
$$T_1 \text{ is tangent to } S_1 \text{ at } P \Rightarrow T_1 : y \cdot 2 = x + 2$$

$$\Rightarrow T_1 : x - 2y + 2 = 0$$

$$T_2 \text{ is tangent to } S_2 \text{ at } P \Rightarrow T_2 : x \cdot 2 + y \cdot 2 = 2(x+2)$$

$$\Rightarrow T_2 : y = 2$$

& $L_3 : x + y + 2 = 0$ is third line



$$PQ = a = \sqrt{20}$$

$$QR = b = \sqrt{8}$$

$$RP = c = 6$$

$$\text{Area } (\Delta PQR) = \Delta = \frac{1}{2} \times 6 \times 2 = 6$$

$$\therefore r = \frac{abc}{4\Delta} = \frac{\sqrt{160}}{4} = \sqrt{10} \Rightarrow r^2 = 10$$

83. A circle with centre $(2, 3)$ and radius 4 intersects the line $x + y = 3$ at the points P and Q. If the tangents at P and Q intersect at the point $S(\alpha, \beta)$, then $4\alpha - 7\beta$ is equal to _____.

Official Ans. by NTA (11)

Allen Ans. (11)

Sol. The given line is polar or $P(2, \beta)$ w.r.t. given circle

$$x^2 + y^2 - 4x - 6y - 3 = 0$$

Chord or contact

$$\alpha x + \beta y - 2(x + \alpha) - 3(y + \beta) - 3 = 0$$

$$\Rightarrow (\alpha - 2)x + (\beta - 3)y - (2\alpha + 3\beta + 3) = 0 \dots\dots (i)$$

\because But the equation of chord of contact is given

$$\text{as : } x + y - 3 = 0 \dots\dots (ii)$$

comparing the coefficients

$$\frac{\alpha - 2}{1} = \frac{\beta - 3}{1} = -\left(\frac{2\alpha + 3\beta + 3}{-3}\right)$$

$$\text{On solving } \alpha = -6$$

$$\beta = -5$$

$$\text{Now } 4\alpha - 7\beta = 11$$

84. Let $a_1 = b_1 = 1$ and $a_n = a_{n-1} + (n - 1)$, $b_n = b_{n-1} + a_{n-1}$, $\forall n \geq 2$. If $S = \sum_{n=1}^{10} \frac{b_n}{2^n}$ and $T = \sum_{n=1}^8 \frac{n}{2^{n-1}}$, then $2^7(2S - T)$ is equal to _____.

Official Ans. by NTA (461)

Allen Ans. (461)

Sol. As, $S = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots\dots + \frac{b_9}{2^9} + \frac{b_{10}}{2^{10}}$

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2^2} + \frac{b_2}{2^3} + \dots\dots + \frac{b_9}{2^{10}} + \frac{b_{10}}{2^{11}}$$

subtracting

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2} + \left(\frac{a_1}{2^2} + \frac{a_2}{2^3} + \dots\dots + \frac{a_9}{2^{10}} \right) - \frac{b_{10}}{2^{11}}$$

$$\Rightarrow S = b_1 - \frac{b_{10}}{2^{10}} + \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \dots\dots + \frac{a_9}{2^9} \right)$$

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2} - \frac{b_{10}}{2^{11}} + \left(\frac{a_1}{2^2} + \frac{a_2}{2^3} + \dots\dots + \frac{a_9}{2^{10}} \right)$$

subtracting

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2} - \frac{b_{10}}{2^{11}} + \left(\frac{a_1}{2} - \frac{a_9}{2^{10}} \right) + \left(\frac{1}{2^2} + \frac{2}{2^3} + \dots + \frac{8}{2^9} \right)$$

$$\Rightarrow \frac{S}{2} = \frac{a_1 + b_1}{2} - \frac{(b_{10} + 2a_9)}{2^{11}} + \frac{T}{4}$$

$$\Rightarrow 2S = 2(a_1 + b_1) - \frac{(b_{10} + 2a_9)}{2^9} + T$$

$$\Rightarrow 2^7 (2S - T) = 2^8 (a_1 + b_1) - \frac{(b_{10} + 2a_9)}{4}$$

Given $a_n - a_{n-1} = n - 1$,

$$\therefore a_2 - a_1 = 1$$

$$a_3 - a_2 = 2$$

⋮

$$a_9 - a_8 = 8$$

$$a_9 - a_1 = 1 + 2 + \dots + 8 = 36$$

$$\Rightarrow a_9 = 37 \quad (a_1 = 1)$$

Also, $b_n - b_{n-1} = a_{n-1}$

$$\therefore b_{10} - b_1 = a_1 + a_2 + \dots + a_9$$

$$= 1 + 2 + 4 + 7 + 11 + 16 + 22 + 29 + 37$$

$$\Rightarrow b_{10} = 130 \quad (\text{As } b_1 = 1)$$

$$\therefore 2^7 (2S - T) = 2^8 (1 + 1) - (130 + 2 \times 37)$$

$$2^9 - \frac{204}{4} = 461$$

85. If the equation of the normal to the curve

$$y = \frac{x-a}{(x+b)(x-2)}$$

at the point $(1, -3)$ is $x - 4y = 13$,

then the value of $a + b$ is equal to _____.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $y = \frac{x-a}{(x+b)(x-2)}$

At point $(1, -3)$,

$$-3 = \frac{1-9}{(1+b)(1-2)}$$

$$\Rightarrow 1 - a = 3(1 + b) \quad \dots \dots (1)$$

Now, $y = \frac{x-a}{(x+b)(x-2)}$

$$\Rightarrow \frac{dy}{dx} = \frac{(x+b)(x-2) \times (1) - (x-a)(2x+b-2)}{(x+b)^2(x-2)^2}$$

At $(1, -3)$ slope of normal is $\frac{1}{4}$ hence $\frac{dy}{dx} = -4$,

$$\text{So, } -4 = \frac{(1+b)(-1) - (1-a)b}{(1+b)^2(-1)^2}$$

Using equation (1)

$$\Rightarrow -4 = \frac{(1+b)(-1) - 3(b+1)b}{(1+b)^2}$$

$$\Rightarrow -4 = \frac{(-1) - 3b}{(1+b)} (b \neq -1)$$

$$\Rightarrow b = -3$$

$$\text{So, } a = 7$$

$$\text{Hence, } a + b = 7 - 3 = 4$$

86. Let A be a symmetric matrix such that $|A| = 2$ and

$$\begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} A = \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$$

elements of A is s , then $\frac{\beta s}{\alpha^2}$ is equal to _____.

Official Ans. by NTA (5)

Allen Ans. (5)

$$\begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$$

$$\text{Now } ac - b^2 = 2 \quad \text{and} \quad 2a + b = 1$$

$$\text{and } 2b + c = 2$$

solving all these above equations we get

$$\frac{1-b}{2} \times \begin{pmatrix} 2-2b \\ 1 \end{pmatrix} - b^2 = 2$$

$$\Rightarrow (1-b)^2 - b^2 = 2$$

$$\Rightarrow 1 - 2b = 2$$

$$\Rightarrow b = -\frac{1}{2} \quad \text{and} \quad a = \frac{3}{4} \quad \text{and} \quad c = 3$$

$$\text{Hence } \alpha = 3a + \frac{3b}{2} = \frac{9}{4} - \frac{3}{4} = \frac{3}{2}$$

$$\text{and } \beta = 3b + \frac{3c}{2} = -\frac{3}{2} + \frac{9}{2} = 3$$

$$\text{also } s = a + c = \frac{15}{4}$$

$$\therefore \frac{\beta s}{\alpha^2} = \frac{3 \times 15}{4 \times \frac{9}{4}} = 5$$

87. Let $\{a_k\}$ and $\{b_k\}$, $k \in \mathbb{N}$, be two G.P.s with common ratio r_1 and r_2 respectively such that $a_1 = b_1 = 4$ and $r_1 < r_2$. Let $c_k = a_k + b_k$, $k \in \mathbb{N}$. If $c_2 = 5$ and $c_3 = \frac{13}{4}$ then $\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$ is equal to _____.

Official Ans. by NTA (9)

Allen Ans. (9)

Sol. Given that

$$\begin{aligned} c_k &= a_k + b_k \text{ and } a_1 = b_1 = 4 \\ \text{also } a_2 &= 4r_1 \quad a_3 = 4r_1^2 \\ b_2 &= 4r_2 \quad b_3 = 4r_2^2 \end{aligned}$$

$$\text{Now } c_2 = a_2 + b_2 = 5 \text{ and } c_3 = a_3 + b_3 = \frac{13}{4}$$

$$\Rightarrow r_1 + r_2 = \frac{5}{4} \text{ and } r_1^2 + r_2^2 = \frac{13}{16}$$

$$\text{Hence } r_1 r_2 = \frac{3}{8} \text{ which gives } r_1 = \frac{1}{2} \quad \& \quad r_2 = \frac{3}{4}$$

$$\begin{aligned} \sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4) &= \frac{4}{1-r_1} + \frac{4}{1-r_2} - \left(\frac{48}{32} + \frac{27}{2} \right) \\ &= 24 - 15 = 9 \end{aligned}$$

88. Let $X = \{11, 12, 13, \dots, 40, 41\}$ and $Y = \{61, 62, 63, \dots, 90, 91\}$ be the two sets of observations. If \bar{x} and \bar{y} are their respective means and σ^2 is the variance of all the observations in $X \cup Y$, then $|\bar{x} + \bar{y} - \sigma^2|$ is equal to _____.

Official Ans. by NTA (603)

Allen Ans. (603)

Sol. $\bar{x} = \frac{\sum_{i=11}^{41} i}{31} = \frac{11+41}{2} = 26$ (31 elements)

$$\bar{y} = \frac{\sum_{j=61}^{91} j}{31} = \frac{61+91}{2} = 76$$
 (31 elements)

$$\text{Combined mean, } \mu = \frac{31 \times 26 + 31 \times 76}{31+31}$$

$$= \frac{26+76}{2} = 51$$

$$\sigma^2 = \frac{1}{62} \times \left(\sum_{i=1}^{31} (x_i - \mu)^2 + \sum_{i=1}^{31} (y_i - \mu)^2 \right) = 705$$

Since, $x_i \in X$ are in A.P. with 31 elements & common difference 1, same is $y_i \in Y$, when written in increasing order.

$$\begin{aligned} \therefore \sum_{i=1}^{31} (x_i - \mu)^2 &= \sum_{i=1}^{31} (y_i - \mu)^2 \\ &= 10^2 + 11^2 + \dots + 40^2 \\ &= \frac{40 \times 41 \times 81}{6} - \frac{9 \times 10 \times 19}{6} = 21855 \\ \therefore |\bar{x} + \bar{y} - \sigma^2| &= |26 + 76 - 705| = 603 \end{aligned}$$

89. Let $\alpha = 8 - 14i$, $A = \left\{ z \in \mathbb{C} : \frac{\alpha z - \bar{\alpha} \bar{z}}{z^2 - (\bar{z})^2 - 112i} = 1 \right\}$

and $B = \left\{ z \in \mathbb{C} : |z + 3i| = 4 \right\}$.

Then $\sum_{z \in A \cap B} (Re z - Im z)$ is equal to _____.

Official Ans. by NTA (14)

Allen Ans. (14)

Sol. $\alpha = 8 - 14i$

$$z = x + iy$$

$$az = (8x + 14y) + i(-14x + 8y)$$

$$z + \bar{z} = 2x \quad z - \bar{z} = 2iy$$

Set A: $\frac{2i(-14x + 8y)}{i(4xy - 112)} = 1$

$$(x - 4)(y + 7) = 0$$

$$x = 4 \quad \text{or} \quad y = -7$$

Set B: $x^2 + (y + 3)^2 = 16$

when $x = 4 \quad y = -3$

when $y = -7 \quad x = 0$

$$\therefore A \cap B = \{4 - 3i, 0 - 7i\}$$

So, $\sum_{z \in A \cap B} (Re z - Im z) = 4 - (-3) + (0 - (-7)) = 14$

- 90.** Let $\alpha_1, \alpha_2, \dots, \alpha_7$ be the roots of the equation $x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_7|$.

Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ is equal to _____.

Official Ans. by NTA (9)

Allen Ans. (9)

Sol. Given equation can be rearranged as

$$x(x^6 + 3x^4 - 13x^2 - 15) = 0$$

clearly $x = 0$ is one of the root and other part can be observed by replacing $x^2 = t$ from which we have

$$t^3 + 3t^2 - 13t - 15 = 0$$

$$\Rightarrow (t - 3)(t^2 + 6t + 5) = 0$$

$$\text{So, } t = 3, t = -1, t = -5$$

Now we are getting $x^2 = 3, x^2 = -1, x^2 = -5$

$$\Rightarrow x = \pm\sqrt{3}, x = \pm i, x = \pm\sqrt{5}i$$

From the given condition $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_7|$

We can clearly say that $|\alpha_7| = 0$ and

$$\text{and } |\alpha_6| = \sqrt{5} = |\alpha_5|$$

$$\text{and } |\alpha_4| = \sqrt{3} = |\alpha_3| \text{ and } |\alpha_2| = 1 = |\alpha_1|$$

So we can have, $\alpha_1 = \sqrt{5}i, \alpha_2 = -\sqrt{5}i, \alpha_3 = \sqrt{3}i,$

$$\alpha_4 = -\sqrt{3}, \alpha_5 = i, \alpha_6 = -i$$

Hence

$$\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$$

$$= 1 - (-3) + 5 = 9$$

