FINAL JEE–MAIN EXAMINATION – JANUARY, 2023

(Held On Sunday 29th January, 2023) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

61. The domain of $f(x)=\dfrac{\log_{(x+1)}(x-2)}{e^{2\log_e x}-(2x+3)}, x\in R$ is

(1) $\mathbb{R}-\{1-3\}$ (2) $(2,\infty)-\{3\}$

(3) $(-1,\infty)-\{3\}$ (4) $\mathbb{R}-\{3\}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $x-2>0 \Rightarrow x>2$

$x+1>0 \Rightarrow x>-1$

$x+1\neq 1 \Rightarrow x\neq 0$ and $x>0$

Denominator

$x^2-2x-3\neq 0$

$(x-3)(x+1)\neq 0$

$x\neq -1, 3$

So Ans $(2,\infty)-\{3\}$

62. Let $f:R\to R$ be a function such that $f(x)=\dfrac{x^2+2x+1}{x^2+1}$. Then

(1) f(x) is many-one in $(-\infty, -1)$

(2) f(x) is many-one in $(1, \infty)$

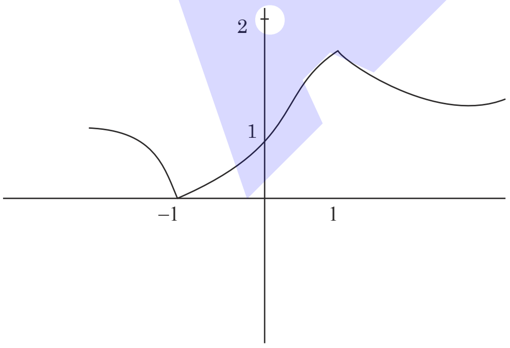
(3) f(x) is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$

(4) f(x) is one-one in $(-\infty, \infty)$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.



$$f(x)=\frac{(x+1)^2}{x^2+1}=1+\frac{2x}{x^2+1}$$

$$f(x)=1+\frac{2}{x+\frac{1}{x}}$$

63. For two non-zero complex number z_1 and z_2, if $\text{Re}(z_1z_2)=0$ and $\text{Re}(z_1+z_2)=0$, then which of the following are possible ?

(A) $\text{Im}(z_1)>0$ and $\text{Im}(z_2)>0$

(B) $\text{Im}(z_1)<0$ and $\text{Im}(z_2)>0$

(C) $\text{Im}(z_1)>0$ and $\text{Im}(z_2)<0$

(D) $\text{Im}(z_1)<0$ and $\text{Im}(z_2)<0$

Choose the correct answer from the options given below :

(1) B and D (2) B and C

(3) A and B (4) A and C

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $z_1=x_1+iy_1$

$z_2=x_2+iy_2$

$\text{Re}(z_1z_2)=x_1x_2-y_1y_2=0$

$\text{Re}(z_1+z_2)=x_1+x_2=0$

x_1 & x_2 are of opposite sign

y_1 & y_2 are of opposite sign

64. Let $\lambda\neq 0$ be a real number. Let α, β be the roots of the equation $14x^2-31x+3\lambda=0$ and α, γ be the roots of the equation $35x^2-53x+4\lambda=0$. Then $\dfrac{3\alpha}{\beta}$ and $\dfrac{4\alpha}{\gamma}$ are the roots of the equation :

(1) $7x^2+245x-250=0$

(2) $7x^2-245x+250=0$

(3) $49x^2-245x+250=0$

(4) $49x^2+245x+250=0$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $14x^2 - 31x + 3\lambda = 0$

$\alpha + \beta = \frac{31}{14}$ ....(1) and $\alpha\beta = \frac{3\lambda}{14}$ ....(2)

$35x^2 - 53x + 4\lambda = 0$

$\alpha + \gamma = \frac{53}{35}$ ....(3) and $\alpha\gamma = \frac{4\lambda}{35}$ ....(4)

$\frac{(2)}{(4)} \Rightarrow \frac{\beta}{\gamma} = \frac{3 \times 35}{4 \times 14} = \frac{15}{8} \Rightarrow \beta = \frac{15}{8}\gamma$

$(1) - (3) \Rightarrow \beta - \gamma = \frac{31}{14} - \frac{53}{35} = \frac{155 - 106}{70} = \frac{7}{10}$

$\frac{15}{8}\gamma - \gamma = \frac{7}{10} \Rightarrow \gamma = \frac{4}{5}$

$\Rightarrow \beta = \frac{15}{8} \times \frac{4}{5} = \frac{3}{2}$

$\Rightarrow \alpha = \frac{31}{14} - \beta = \frac{31}{14} - \frac{3}{2} = \frac{5}{7}$

$\Rightarrow \lambda = \frac{14}{3}\alpha\beta = \frac{14}{3} \times \frac{5}{7} \times \frac{3}{2} = 5$

so, sum of roots $\frac{3\alpha}{\beta} + \frac{4\alpha}{\gamma} = \left(\frac{3\alpha\gamma + 4\alpha\beta}{\beta\gamma}\right)$

$= \frac{\left(3 \times \frac{4\lambda}{35} + 4 \times \frac{3\lambda}{14}\right)}{\beta\gamma} = \frac{12\lambda(14 + 35)}{14 \times 35\beta\gamma}$

$= \frac{49 \times 12 \times 5}{490 \times \frac{3}{2} \times \frac{4}{5}} = 5$

Product of roots

$= \frac{3\alpha}{\beta} \times \frac{4\alpha}{\gamma} = \frac{12\alpha^2}{\beta\gamma} = \frac{12 \times \frac{25}{49}}{\frac{3}{2} \times \frac{4}{5}} = \frac{250}{49}$

So, required equation is $x^2 - 5x + \frac{250}{49} = 0$

$\Rightarrow 49x^2 - 245x + 250 = 0$

65. Consider the following system of questions

$\alpha x + 2y + z = 1$

$2\alpha x + 3y + z = 1$

$3x + \alpha y + 2z = \beta$

For some $\alpha, \beta \in \mathbb{R}$. Then which of the following is NOT correct.

(1) It has no solution if $\alpha = -1$ and $\beta \neq 2$

(2) It has no solution for $\alpha = -1$ and for all $\beta \in \mathbb{R}$

(3) It has no solution for $\alpha = 3$ and for all $\beta \neq 2$

(4) It has a solution for all $\alpha \neq -1$ and $\beta = 2$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $D = \begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & 2 \end{vmatrix} = 0 \Rightarrow \alpha = -1, 3$

$D_x = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 1 \\ \alpha & 2 & \beta \end{vmatrix} = 0 \Rightarrow \beta = 2$

$D_y = \begin{vmatrix} \alpha & 1 & 1 \\ 2\alpha & 1 & 1 \\ 3 & 2 & \beta \end{vmatrix} = 0$

$D_z = \begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & \beta \end{vmatrix} = 0$

$\beta = 2, \alpha = -1$

$\alpha = -1, \beta = 2$ Infinite solution

66. Let α and β be real numbers. Consider a 3 × 3 matrix A such that $A^2 = 3A + \alpha I$. If $A^4 = 21A + \beta I$, then

(1) $\alpha = 1$ (2) $\alpha = 4$

(3) $\beta = 8$ (4) $\beta = -8$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $A^2 = 3A + \alpha I$

$A^3 = 3A^2 + \alpha A$

$A^3 = 3(3A + \alpha I) + \alpha A$

$A^3 = 9A + \alpha A + 3\alpha I$

$A^4 = (9+\alpha)A^2 + 3\alpha A$

$= (9+\alpha)(3A + \alpha I) + 3\alpha A$

$= A(27 + 6\alpha) + \alpha(9+\alpha)$

$\Rightarrow 27 + 6\alpha = 21 \Rightarrow \alpha = -1$

$\Rightarrow \beta = \alpha(9+\alpha) = -8$

67. Let x = 2 be a root of the equation $x^2 + px + q = 0$

and $f(x) = \begin{cases} \dfrac{1-\cos(x^2-4px+q^2+8q+16)}{(x-2p)^4}, & x \neq 2p \\ 0, & x = 2p \end{cases}$

Then $\lim\limits_{x\to 2p^+}[f(x)]$

where [.] denotes greatest integer function, is

(1) 2 (2) 1

(3) 0 (4) –1

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

$$\lim_{x\to 2p^+}\left(\frac{1-\cos\left(x^2-4px+q^2+8q+16\right)}{\left(x^2-4px+q^2+8q+16\right)^2}\right)\left(\frac{\left(x^2-4px+q^2+8q+16\right)^2}{(x-2p)^2}\right)$$

$$\lim_{h\to 0}\frac{1}{2}\left(\frac{(2p+h)^2-4p(2p+h)+q^2+82+16}{h^2}\right)^2 = \frac{1}{2}$$

Using L'Hospital's

$$\lim_{x\to 2p^+}[f(x)] = 0$$

68. Let $f(x) = x + \dfrac{a}{\pi^2-4}\sin x + \dfrac{b}{\pi^2-4}\cos x$, $x \in \mathbb{R}$ be a function which satisfies $f(x) = x + \int\limits_0^{\pi/2}\sin(x+y)\,f(y)\,dy$. Then (a + b) is equal to

(1) $-\pi(\pi+2)$ (2) $-2\pi(\pi+2)$

(3) $-2\pi(\pi-2)$ (4) $-\pi(\pi-2)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $f(x) = x + \int\limits_0^{\pi/2}(\sin x\cos y + \cos x\sin y)f(y)\,dy$

$f(x) = x + \int\limits_0^{\pi/2}\left((\cos y\ f(y)dy)\sin x + (\sin y\ f(y)dy)\cos x\right)$ **...(1)**

On comparing with

$f(x) = x + \dfrac{a}{\pi^2-4}\sin x + \dfrac{b}{\pi^2-4}\cos x$, $x\in\mathbb{R}$ then

$\Rightarrow \dfrac{a}{\pi^2-4} = \int\limits_0^{\pi/2}\cos y\ f(y)\,dy$ **...(2)**

$\Rightarrow \dfrac{b}{\pi^2-4} = \int\limits_0^{\pi/2}\sin y\ f(y)\,dy$ **...(3)**

Add (2) and (3)

$\dfrac{a+b}{\pi^2-4} = \int\limits_0^{\pi/2}(\sin y + \cos y)f(y)\,dy$ ......(4)

$\dfrac{a+b}{\pi^2-4} = \int\limits_0^{\pi/2}(\sin y + \cos y)f\left(\dfrac{\pi}{2}-y\right)dy$ ...(5)

Add (4) and (5)

$\dfrac{2(a+b)}{\pi^2-4} = \int\limits_0^{\pi/2}(\sin y+\cos y)\left(\dfrac{\pi}{2} + \dfrac{(a+b)}{\pi^2-4}(\sin y + \cos y)\right)dy$

$= \pi + \dfrac{a+b}{\pi^2-4}\left(\dfrac{\pi}{2}+1\right)$

$(a+b) = -2\pi(\pi+2)$

69. Let $A = \left\{(x,y)\in\mathbb{R}^2 : y \geq 0, 2x \leq y \leq \sqrt{4-(x-1)^2}\right\}$

and $B = \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : 0 \leq y \leq \min\left\{2x, \sqrt{4-(x-1)^2}\right\}\right\}$

Then the ratio of the area of A to the area of B is

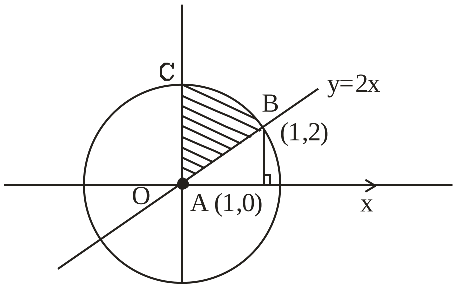
(1) $\dfrac{\pi-1}{\pi+1}$ (2) $\dfrac{\pi}{\pi-1}$

(3) $\dfrac{\pi}{\pi+1}$ (4) $\dfrac{\pi+1}{\pi-1}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $y^2+(x-1)^2=4$

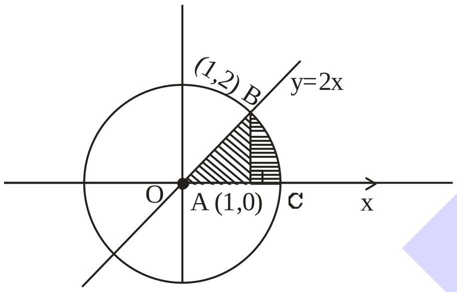


shaded portion = circular (OABC)

$-\text{Ar}(\Delta OAB)$

$=\frac{\pi(4)}{4}-\frac{1}{2}(2)(1)$

$A=(\pi-1)$



Area B = Ar (ΔAOB) + Area of arc of circle (ABC)

$=\frac{1}{2}(1)(2)+\frac{\pi(2)^2}{4}=\pi+1$

$\frac{A}{B}=\frac{\pi-1}{\pi+1}$

70. Let Δ be the area of the region

$\{(x,y)\in\mathbb{R}^2: x^2+y^2\le 21, y^2\le 4x, x\ge 1\}$. Then

$\frac{1}{2}\left(\Delta-21\sin^{-1}\frac{2}{\sqrt{7}}\right)$ is equal to

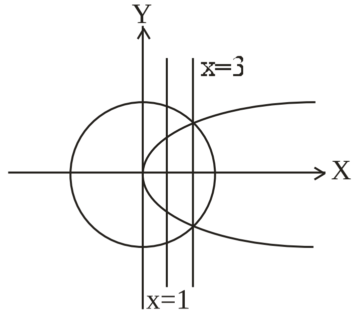
(1) $2\sqrt{3}-\frac{1}{3}$ (2) $\sqrt{3}-\frac{2}{3}$

(3) $2\sqrt{3}-\frac{2}{3}$ (4) $\sqrt{3}-\frac{4}{3}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



Area $2\int_1^3 2\sqrt{x}\,dx+2\int_3^{\sqrt{21}}\sqrt{21-x^2dx}$

$\Delta=\frac{8}{3}(3\sqrt{3}-1)+21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)-6\sqrt{3}$

$\frac{1}{2}\left(\Delta-21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right)=\frac{2\sqrt{3}-\frac{8}{3}}{2}$

$=\sqrt{3}-\frac{4}{3}$

71. A light ray emits from the origin making an angle 30º with the positive x-axis. After getting reflected by the line x + y = 1, if this ray intersects x-axis at Q, then the abscissa of Q is

(1) $\frac{2}{(\sqrt{3}-1)}$ (2) $\frac{2}{3+\sqrt{3}}$

(3) $\frac{2}{3-\sqrt{3}}$ (4) $\frac{\sqrt{3}}{2(\sqrt{3}+1)}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Slope of reflected ray = $\tan 60^\circ=\sqrt{3}$

Line $y=\frac{x}{\sqrt{3}}$ intersect $y+x=1$ at $\left(\frac{\sqrt{3}}{\sqrt{3}+1},\frac{1}{\sqrt{3}+1}\right)$

Equation of reflected ray is

$y-\frac{1}{\sqrt{3}+1}=\sqrt{3}\left(x-\frac{\sqrt{3}}{\sqrt{3}+1}\right)$

Put $y=0\Rightarrow x=\frac{2}{3+\sqrt{3}}$

72. Let B and C be the two points on the line y + x = 0 such that B and C are symmetric with respect to the origin. Suppose A is a point on y – 2x = 2 such that ΔABC is an equilateral triangle. Then, the area of the ΔABC is

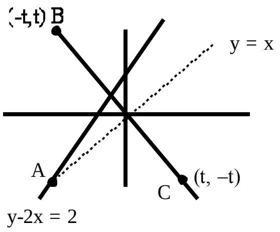
(1) $3\sqrt{3}$ (2) $2\sqrt{3}$

(3) $\frac{8}{\sqrt{3}}$ (4) $\frac{10}{\sqrt{3}}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.



At A x = y

Y – 2x = 2

(–2, –2)

Height from line x + y = 0

$h = \frac{4}{\sqrt{2}}$

Area of $\Delta = \frac{\sqrt{3}}{4}\frac{h^2}{\sin^2 60} = \frac{8}{\sqrt{3}}$

73. Let the tangents at the points A (4, –11) and B(8, –5) on the circle $x^2 + y^2 - 3x + 10y - 15 = 0$, intersect at the point C. Then the radius of the circle, whose centre is C and the line joining A and B is its tangent, is equal to

(1) $\frac{3\sqrt{3}}{4}$ (2) $2\sqrt{13}$

(3) $\sqrt{13}$ (4) $\frac{2\sqrt{13}}{3}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Equation of tangent at A (4, –11) on circle is

$$\Rightarrow 4x - 11y - 3\left(\frac{x+4}{2}\right) + 10\left(\frac{y-11}{2}\right) - 15 = 0$$

$$\Rightarrow 5x - 12y - 152 = 0 \quad \ldots\ldots (1)$$

Equation of tangent at B (8, –5) on circle is

$$\Rightarrow 8x - 5y - 3\left(\frac{x+8}{2}\right) + 10\left(\frac{y-5}{2}\right) - 15 = 0$$

$$\Rightarrow 13x - 104 = 0 \Rightarrow x = 8$$

put in (1) $\Rightarrow y = \frac{28}{3}$

$$r = \left|\frac{3.8 + \frac{2.28}{3} - 34}{\sqrt{13}}\right| = \frac{2\sqrt{13}}{3}$$

74. Let [x] denote the greatest integer $\le x$. Consider the function f(x) = max $\{x^2, 1+[x]\}$. Then the value of the integral $\int_0^2 f(x)dx$ is :

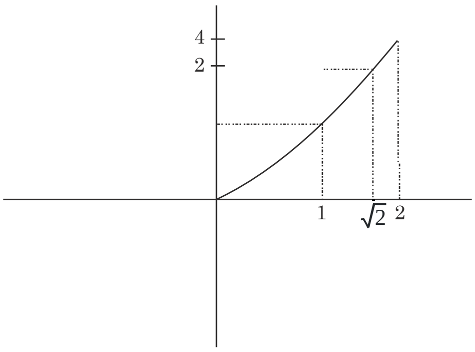
(1) $\frac{5+4\sqrt{2}}{3}$ (2) $\frac{8+4\sqrt{2}}{3}$

(3) $\frac{1+5\sqrt{2}}{3}$ (4) $\frac{4+5\sqrt{2}}{3}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.



$$A = \int_0^1 1.dx + \int_1^{\sqrt{2}} 2dx + \int_{\sqrt{2}}^2 x^2 dx$$

$$= 1 + 2\sqrt{2} - 2 + \frac{8}{3} - \frac{2\sqrt{2}}{3}$$

$$= \frac{5}{3} + \frac{4\sqrt{2}}{3}$$

75. If the vectors $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$ are coplanar and the projection of $\vec{a}$ on the vector $\vec{b}$ is $\sqrt{54}$ units, then the sum of all possible values of $\lambda + \mu$ is equal to

(1) 0 (2) 6

(3) 24 (4) 18

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\begin{vmatrix} \lambda & \mu & 4 \\ -2 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 0$

$\lambda(10) - \mu(2) + 4(-14) = 0$

$10\lambda - 2\mu = 56$

$5\lambda - \mu = 28$ .....(1)

$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \sqrt{54}$

$\frac{-2\lambda + 4\mu - 8}{\sqrt{24}} = \sqrt{54}$

$-2\lambda + 4\mu - 8 = \sqrt{54 \times 24}$ ....(2)

By solving equation (1) & (2)

$\Rightarrow \lambda + \mu = 24$

76. Fifteen football players of a club-team are given 15 T-shirts with their names written on the backside. If the players pick up the T-shirts randomly, then the probability that at least 3 players pick the correct T-shirt is

(1) $\frac{5}{24}$ (2) $\frac{2}{15}$

(3) $\frac{1}{6}$ (4) $\frac{5}{36}$

Official Ans. by NTA (DROP)

Sol.

Required probability $= 1 - \frac{D_{(15)} + {}^{15}C_1 . D_{(14)} + {}^{15}C_2 D_{(13)}}{15!}$

Taking $D_{(15)}$ as $\frac{15!}{e}$

$D_{(14)}$ as $\frac{14!}{e}$

$D_{(13)}$ as $\frac{13!}{e}$

We get, $1 - \left(\frac{\frac{15!}{e} + 15.\frac{14!}{e} + \frac{15\times14}{2}\times\frac{13!}{e}}{15!}\right)$

$= 1 - \left(\frac{1}{e} + \frac{1}{e} + \frac{1}{2e}\right) = 1 - \frac{5}{2e} \approx .08$

77. Let $f(\theta) = 3\left(\sin^4\left(\frac{3\pi}{2} - \theta\right) + \sin^4(3\pi + \theta)\right) - 2(1 - \sin^2 2\theta)$ and

$S = \left\{\theta \in [0,\pi] : f'(\theta) = -\frac{\sqrt{3}}{2}\right\}$. If $4\beta = \sum_{\theta\in S}\theta$,

then $f(\beta)$ is equal to

(1) $\frac{11}{8}$ (2) $\frac{5}{4}$

(3) $\frac{9}{8}$ (4) $\frac{3}{2}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

$f(\theta) = 3\left(\sin^4\left(\frac{3\pi}{2} - \theta\right) + \sin^4(3x + \theta)\right) - 2(1 - \sin^2 2\theta)$

$S = \left\{\theta \in [0,\pi] : f'(\theta) = -\frac{\sqrt{3}}{2}\right\}$

$\Rightarrow f(\theta) = 3(\cos^4\theta + \sin^4\theta) - 2\cos^2 2\theta$

$\Rightarrow f(\theta) = 3\left(1 - \frac{1}{2}\sin^2 2\theta\right) - 2\cos^2 2\theta$

$\Rightarrow f(\theta) = 3 - \frac{3}{2}\sin^2 2\theta - 2\cos^2\theta$

$= \frac{3}{2} - \frac{1}{2}\cos^2 2\theta = \frac{3}{2} - \frac{1}{2}\left(\frac{1 + \cos 4\theta}{2}\right)$

$f(\theta) = \frac{5}{4} - \frac{\cos 4\theta}{4}$

$f'(\theta) = \sin 4\theta$

$\Rightarrow f'(\theta) = \sin 4\theta = -\frac{\sqrt{3}}{2}$

$\Rightarrow 4\theta = n\pi + (-1)^n \frac{\pi}{3}$

$\Rightarrow \theta = \frac{n\pi}{4} + (-1)^n \frac{\pi}{12}$

$$\Rightarrow \theta=\frac{\pi}{12},\left(\frac{\pi}{4}-\frac{\pi}{12}\right),\left(\frac{\pi}{2}+\frac{\pi}{12}\right),\left(\frac{3\pi}{4}-\frac{\pi}{12}\right)$$

$$\Rightarrow 4\beta=\frac{\pi}{4}+\frac{\pi}{2}+\frac{3\pi}{4} = \frac{3\pi}{2}$$

$$\Rightarrow \beta=\frac{3\pi}{8}\Rightarrow f(\beta)=\frac{5}{4}-\frac{\cos\frac{3\pi}{2}}{4}=\frac{5}{4}$$

78. If p, q and r are three propositions, then which of the following combination of truth values of p, q and r makes the logical expression $\{(p\vee q)\wedge((\sim p)\vee r)\}\rightarrow((\sim q)\vee r)$ false ?

(1) p = T, q = F, r = T (2) p = T, q = T, r = F

(3) p = F, q = T, r = F (4) p = T, q = F, r = F

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

	p	q	r	$(p\vee q)\wedge((\sim p)\vee r)$	$\sim q\vee r$
(1)	T	F	T	T	T
(2)	T	T	F	F	F
(3)	F	T	F	T	F
(4)	T	F	F	F	T

Option (3) $(p\vee q)\wedge(\sim q\vee r)\rightarrow(\sim p\vee r)$ will be False.

79. There rotten apples are mixed accidently with seven good apples and four apples are drawn one by one without replacement. Let the random variable X denote the number of rotten apples. If μ and σ^2 represent mean and variance of X, respectively, then $10(\mu^2+\sigma^2)$ is equal to

(1) 20

(2) 250

(3) 25

(4) 30

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

X	P(x)	XP(X)	$X^2P(X)$
0	1/6	0	0
1	1/2	1/2	1/2
2	3/10	6/10	12/10
3	1/30	1/10	9/30

$$\sum xP(x)=\frac{6}{2}=\mu$$

$$\sigma^2=\sum x^2P(x)-\mu^2$$

$$\sigma^2+\mu^2=0+\frac{1}{2}+\frac{12}{10}+\frac{9}{30}=2$$

$$10\left(\sigma^2+\mu^2\right)=20 \text{ Ans.}$$

80. Let y = f(x) be the solution of the differential equation $y(x+1)\,dx - x^2dy = 0$, $y(1) = e$. Then $\lim\limits_{x\to 0^+} f(x)$ is equal to

(1) 0 (2) $\frac{1}{e}$

(3) e^2 (4) $\frac{1}{e^2}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\frac{x+1}{x^2}dx=\frac{dy}{y}$

$$\ln x-\frac{1}{x}=\ln y+c$$

(1, e)

c = -2

$$\ln x-\frac{1}{x}=\ln y-2$$

$$y=e^{\ln x}-\frac{1}{x}+2$$

$$\lim_{x\to 0^+} e^{\ln x-1}-\frac{1}{x}+2$$

$$=e^{-\infty}$$

$$=0$$

SECTION-B

81. Let the co-ordinates of one vertex of ΔABC be $A(0, 2, \alpha)$ and the other two vertices lie on the line $\frac{x+\alpha}{5}=\frac{y-1}{2}=\frac{z+4}{3}$. For $\alpha \in \mathbb{Z}$, if the area of ΔABC is 21 sq. units and the line segment BC has length $2\sqrt{21}$ units, then α^2 is equal to _____.

Official Ans. by NTA (9)

Allen Ans. (9)

Sol. A. $(O_1\ 2,\ \alpha)$

$$\left|\frac{1}{2}.2\sqrt{21}.\left|\begin{matrix} i & j & k \\ \alpha & 1 & \alpha+4 \\ 5 & 2 & 3 \end{matrix}\right|\frac{1}{\sqrt{25+4+9}}\right| = 21\sqrt{21}$$

$$\sqrt{(2\alpha+5)^2+(2\alpha+20)^2+(2\alpha-5)^2} = \sqrt{21}\sqrt{38}$$

$$\Rightarrow 12\alpha^2+80\alpha+450 = 798$$

$$\Rightarrow 12\alpha^2+80\alpha-348=0$$

$$\Rightarrow \alpha=3 \Rightarrow \alpha^2=9$$

82. Let the equation of the plane P containing the line $x+10=\frac{8-y}{2}=z$ be $ax + by + 3z = 2(a+b)$ and the distance of the plane P from the point (1, 27, 7) be c. Then $a^2 + b^2 + c^2$ is equal to ____.

Official Ans. by NTA (355)

Allen Ans. (355)

Sol. The line $\frac{x+10}{1}=\frac{y-8}{-2}=\frac{z}{1}$ have a point (–10, 8, 0) with d. r. (1, –2, 1)

$\because$ the plane $ax + by + 3z = 2(a + b)$

$\Rightarrow b = 2a$

& dot product of d.r.'s is zero

$\therefore a - 2b + 3 = 0$

$\therefore a = 1$ & $b = 2$

Distance from (1, 27, 7) is

$$c=\frac{1+54+21-6}{\sqrt{14}}=\frac{70}{\sqrt{14}}=5\sqrt{14}$$

$\therefore a^2 + b^2 + c^2 = 1 + 4 + 350$

$= 355$

83. Suppose f is a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{N}$ and $f(1)=\frac{1}{5}$. If $\sum_{n=1}^{m}\frac{f(n)}{n(n+1)(n+2)}=\frac{1}{12}$, then m is equal to_____.

Official Ans. by NTA (10)

Allen Ans. (10)

Sol. $\because f(1)=\frac{1}{5}\ \therefore f(2)=f(1)+f(1)=\frac{2}{5}$

$f(2)=\frac{2}{5}\quad f(3)=f(2)+f(1)=\frac{3}{5}$

$f(3)=\frac{3}{5}$

$$\therefore \sum_{n=1}^{m}\frac{f(n)}{n(n+1)(n+2)}$$

$$=\frac{1}{5}\sum_{n=1}^{m}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=\frac{1}{5}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+....+\frac{1}{m+1}-\frac{1}{m+2}\right)$$

$$=\frac{1}{5}\left(\frac{1}{2}-\frac{1}{m+2}\right)=\frac{m}{10(m+2)}=\frac{1}{12}$$

$\therefore m=10$

84. Let a_1, a_2, a_3, ... be a GP of increasing positive numbers. If the product of fourth and sixth terms is 9 and the sum of fifth and seventh terms is 24, then $a_1a_9 + a_2a_4a_9 + a_5 + a_7$ is equal to ____.

Official Ans. by NTA (60)

Allen Ans. (60)

Sol. $a_4 . a_6 = 9 \Rightarrow (a_5)^2=9 \Rightarrow a_5=3$

& $a_5+a_7=24 \Rightarrow a_5+a_5r^2=24 \Rightarrow (1+r^2)=8 \Rightarrow r=\sqrt{7}$

$\Rightarrow a=\frac{3}{49}$

$\Rightarrow a_1a_9+a_2a_4a_9+a_5+a_7=9+27+3+21=60$

85. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero non-coplanar vectors. Let the position vectors of four points A, B, C and D be $\vec{a}-\vec{b}+\vec{c}$, $\lambda\vec{a}-3\vec{b}+4\vec{c}$, $-\vec{a}+2\vec{b}-3\vec{c}$ and $2\vec{a}-4\vec{b}+6\vec{c}$ respectively. If $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then λ is :

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\overline{AB} = (\lambda-1)\bar{a} - 2\bar{b} + 3\bar{c}$

$\overline{AC} = 2\bar{a} + 3\bar{b} - 4\bar{c}$

$\overline{AD} = \bar{a} - 3\bar{b} + 5\bar{c}$

$$\begin{vmatrix} \lambda-1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$$

$\Rightarrow (\lambda-1)(15-12)+2(-10+4)+3(6-3)=0$

$\Rightarrow (\lambda-1)=1 \Rightarrow \lambda = 2$

86. If all the six digit numbers $x_1\ x_2\ x_3\ x_4\ x_5\ x_6$ with $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are arranged in the increasing order, then the sum of the digits in the 72^{th} number is _____.

Official Ans. by NTA (32)

Allen Ans. (32)

Sol.

1	2					$= {}^7C_4 = 35$
1	3					$= {}^6C_4 = 15$
1	4					$= {}^5C_4 = 5$
1	5					$= {}^4C_4 = 1$
2	3					$= {}^6C_4 = 15$

71 words

2 4 5 6 7 8 $\rightarrow$ 72^{th} word

$2+4+5+6+7+8=32$

87. Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable function that satisfies the relation $f(x+y) = f(x) + f(y) - 1$, $\forall x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to _____.

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $f(x+y) = f(x) + f(y) - 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = f'(0) = 2$$

$f'(x) = 2 \Rightarrow dy = 2dx$

$y = 2x + C$

$x = 0,\ y = 1,\ c = 1$

$y = 2x + 1$

$|f(-2)| = |-4+1| = |-3| = 3$

88. If the co-efficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ and the co-efficient of x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ are equal, then $(\alpha\beta)^2$ is equal to ____.

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Coefficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right) = {}^{11}C_6 \cdot \frac{\alpha^5}{\beta^6}$

$\because$ Both are equal

$$\therefore {}^{11}C_6 \cdot \frac{\alpha^5}{\beta^6} = -{}^{11}C_5 \cdot \frac{\alpha^6}{\beta^5}$$

$\Rightarrow \frac{1}{\beta} = -\alpha$

$\Rightarrow \alpha\beta = -1$

$\Rightarrow (\alpha\beta)^2 = 1$

89. Let the coefficients of three consecutive terms in the binomial expansion of $(1+2x)^n$ be in the ratio 2 : 5 : 8. Then the coefficient of the term, which is in the middle of these three terms, is_______.

Official Ans. by NTA (1120)

Allen Ans. (1120)

Sol. $t_{r+1} = {}^nC_r(2x)^r$

$$\Rightarrow \frac{{}^nC_{r-1}(2)^{r-1}}{{}^nC_r(2)^r} = \frac{2}{5}$$

$$\Rightarrow \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!(2)}{r!(n-r)!}} = \frac{2}{5}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{4}{5} \Rightarrow 5r = 4n - 4r + 4$$

$$\Rightarrow 9r = 4(n+1) \qquad \dots (1)$$

$$\Rightarrow \frac{{}^nC_r(2)^r}{{}^nC_{r+1}(2)^{r+1}} = \frac{5}{8}$$

$$\Rightarrow \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r+1)!(n-r-1)!}} = \frac{5}{4} \Rightarrow \frac{r+1}{n-r} = \frac{5}{4}$$

$$\Rightarrow 4r + 4 = 5n - 5r \Rightarrow 5n - 4 = 9r \quad \dots (2)$$

From (1) and (2)

$$\Rightarrow 4n + 4 = 5n - 4 \Rightarrow n = 8$$

(1) $\Rightarrow$ $r = 4$

so, coefficient of middle term is

$$^8C_4\, 2^4 = 16 \times \frac{8\times7\times6\times5}{4\times3\times2\times1} = 16 \times 70 = 1120$$

90. Five digit numbers are formed using the digits 1, 2, 3, 5, 7 with repetitions and are written in descending order with serial numbers. For example, the number 77777 has serial number 1. Then the serial number of 35337 is_____.

Official Ans. by NTA (1436)

Allen Ans. (1436)

Sol. No of 5 digit numbers starting with digit 1

$= 5 \times 5 \times 5 \times 5 = 625$

No of 5 digit numbers starting with digit 2

$= 5 \times 5 \times 5 \times 5 = 625$

No of 5 digit numbers starting with 31

$= 5 \times 5 \times 5 = 125$

No of 5 digit numbers starting with 32

$= 5 \times 5 \times 5 = 125$

No of 5 digit numbers starting with 33

$= 5 \times 5 \times 5 = 125$

No of 5 digit numbers starting with 351

$= 5 \times 5 = 25$

No of 5 digit numbers starting with 352

$= 5 \times 5 = 25$

No of 5 digit numbers starting with 3531

$= 5$

No of 5 digit numbers starting with 3532

$= 5$

Before 35337 will be 4 numbers,

So rank of 35337 will be 1690

So, in descending order serial number will be

3125 – 1690 + 1 = 1436