

FINAL JEE-MAIN EXAMINATION – APRIL, 2024

(Held On Tuesday 09th April, 2024)

TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

1. $\lim_{x \rightarrow 0} \frac{e - (1+2x)^{\frac{1}{2x}}}{x}$ is equal to :

- (1) e (2) $\frac{-2}{e}$
 (3) 0 (4) $e - e^2$

Ans. (1)

Sol. $\lim_{x \rightarrow 0} \frac{e - e^{\frac{1}{2x} \ln(1+2x)}}{x}$

$$= \lim_{x \rightarrow 0} (-e) \frac{\left(e^{\frac{\ln(1+2x)}{2x} - 1} \right)}{x}$$

$$= \lim_{x \rightarrow 0} (-e) \frac{\ln(1+2x) - 2x}{2x^2}$$

$$= (-e) \times (-1) \frac{4}{2 \times 2} = e$$

2. Consider the line L passing through the points (1, 2, 3) and (2, 3, 5). The distance of the point $\left(\frac{11}{3}, \frac{11}{3}, \frac{19}{3}\right)$ from the line L along the line

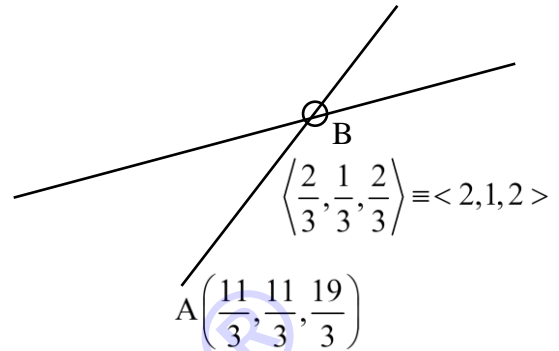
$$\frac{3x-11}{2} = \frac{3y-11}{1} = \frac{3z-19}{2} \text{ is equal to :}$$

- (1) 3 (2) 5
 (3) 4 (4) 6

Ans. (1)

Sol. $\frac{x-1}{2-1} = \frac{y-2}{3-2} = \frac{z-3}{5-3}$

$$\Rightarrow \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{2} = \lambda$$



$$B(1 + \lambda, 2 + \lambda, 3 + 2\lambda)$$

$$\text{D.R. of AB} = \left\langle \frac{3\lambda - 8}{3}, \frac{3\lambda - 5}{3}, \frac{6\lambda - 10}{3} \right\rangle$$

$$B \left(\frac{11}{3}, \frac{11}{3}, \frac{19}{3} \right) \frac{3\lambda - 8}{3\lambda - 5} = \frac{2}{1} \Rightarrow 3\lambda - 8 = 6\lambda - 10$$

$$3\lambda = 2$$

$$\lambda = \frac{2}{3}$$

$$AB = \frac{\sqrt{36+9+36}}{3} = \frac{9}{3} = 3$$

3. Let $\int_0^x \sqrt{1 - (y'(t))^2} dt = \int_0^x y(t) dt, 0 \leq x \leq 3, y \geq 0,$

$y(0) = 0.$ Then at $x = 2,$ $y'' + y + 1$ is equal to :

- (1) 1 (2) 2
 (3) $\sqrt{2}$ (4) $1/2$

Ans. (1)

Sol. $\sqrt{1 - (y'(x))^2} = y(x)$

$$1 - \left(\frac{dy}{dx} \right)^2 = y^2$$

$$\left(\frac{dy}{dx} \right)^2 = 1 - y^2$$



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$$\frac{dy}{\sqrt{1-y^2}} = dx \text{ OR } \frac{dy}{\sqrt{1-y^2}} = -dx$$

$$\Rightarrow \sin^{-1}y = x + c, \sin^{-1}y = -x + c$$

$$x = 0, y = 0 \Rightarrow c = 0$$

$$\sin^{-1}y = x, \text{ as } y \geq 0$$

$$\sin x = y$$

$$\Rightarrow \frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} = -\sin x$$

$$\Rightarrow -\sin x + \sin x + 1 = 1$$

4. Let z be a complex number such that the real part of $\frac{z-2i}{z+2i}$ is zero. Then, the maximum value of $|z-(6+8i)|$ is equal to :

- (1) 12 (2) ∞
(3) 10 (4) 8

Ans. (1)

Sol. $\frac{z-2i}{z+2i} + \frac{\bar{z}+2i}{\bar{z}-2i} = 0$

$$z\bar{z} - 2i\bar{z} - 2iz + 4(-1)$$

$$+z\bar{z} + 2zi + 2\bar{z}i + 4(-1) = 0$$

$$\Rightarrow 2|z|^2 = 8 \Rightarrow |z| = 2$$

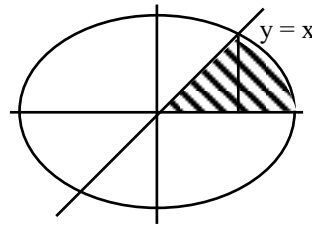
$$|z-(6+8i)|_{\text{maximum}} = 10 + 2 = 12$$

5. The area (in square units) of the region enclosed by the ellipse $x^2 + 3y^2 = 18$ in the first quadrant below the line $y = x$ is :

- (1) $\sqrt{3}\pi + \frac{3}{4}$ (2) $\sqrt{3}\pi$
(3) $\sqrt{3}\pi - \frac{3}{4}$ (4) $\sqrt{3}\pi + 1$

Ans. (2)

Sol. $\frac{x^2}{18} + \frac{y^2}{6} = 1$



$$\frac{x^2}{18} + \frac{3x^2}{18} = 1 \Rightarrow 4x^2 = 18 \Rightarrow x^2 = \frac{9}{2}$$

$$\int_{\frac{3}{\sqrt{2}}}^{3\sqrt{2}} \frac{\sqrt{18-x^2}}{\sqrt{3}} dx$$

$$= \frac{1}{\sqrt{3}} \left(\frac{x\sqrt{18-x^2}}{2} + \frac{18}{2} \sin^{-1} \frac{x}{3\sqrt{2}} \right)_{\frac{3}{\sqrt{2}}}^{3\sqrt{2}}$$

$$= \frac{1}{\sqrt{3}} \left(9 \times \frac{\pi}{2} - \frac{3}{2\sqrt{2}} \times \frac{3\sqrt{3}}{\sqrt{2}} - 9 \times \frac{\pi}{6} \right)$$

Required Area

$$= \frac{1}{2} \times \frac{9}{2} + \left(\frac{18\pi}{6} - \frac{9\sqrt{3}}{4} \right) \frac{1}{\sqrt{3}}$$

$$= \sqrt{3}\pi$$

6. Let the foci of a hyperbola H coincide with the foci of the ellipse $E : \frac{(x-1)^2}{100} + \frac{(y-1)^2}{75} = 1$ and the eccentricity of the hyperbola H be the reciprocal of the eccentricity of the ellipse E . If the length of the transverse axis of H is α and the length of its conjugate axis is β , then $3\alpha^2 + 2\beta^2$ is equal to :

- (1) 242
(2) 225
(3) 237
(4) 205

Ans. (2)



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$$\text{Var}(x) = \frac{c^2(2+2^2+3^2+4^2+5^2+6^2)}{7}$$

$$-\left(\frac{22c}{7}\right)^2$$

$$= \frac{92c^2}{7} - c^2 \times \frac{484}{49}$$

$$= \frac{(644-484)c^2}{49} = \frac{160c^2}{49}$$

$$160 = \frac{160 \times c^2}{49} \Rightarrow c = 7$$

9. Let the range of the function

$$f(x) = \frac{1}{2 + \sin 3x + \cos 3x}, x \in \mathbb{R} \text{ be } [a, b].$$

If α and β are respectively the A.M. and the G.M.

of a and b , then $\frac{\alpha}{\beta}$ is equal to :

(1) $\sqrt{2}$ (2) 2

(3) $\sqrt{\pi}$ (4) π

Ans. (1)

Sol. $f(x) = \frac{1}{2 + \sin 3x + \cos 3x}$

$$\left[\frac{1}{2 + \sqrt{2}}, \frac{1}{2 - \sqrt{2}} \right]$$

$$\frac{\alpha}{\beta} = \frac{a+b}{2\sqrt{ab}} = \frac{1}{2} \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right)$$

$$= \frac{1}{2} \left(\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} + \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} \right)$$

$$= \frac{(2-\sqrt{2}) + (2+\sqrt{2})}{2 \times \sqrt{2}} = \sqrt{2}$$

10. Between the following two statements :

Statement-I : Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and

$\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$. Then the vector \vec{r} satisfying

$\vec{a} \times \vec{r} = \vec{a} \times \vec{b}$ and $\vec{a} \cdot \vec{r} = 0$ is of magnitude $\sqrt{10}$.

Statement-II : In a triangle ABC, $\cos 2A + \cos 2B$
 $+ \cos 2C \geq -\frac{3}{2}$.

- (1) Both Statement-I and Statement-II are incorrect
 (2) Statement-I is incorrect but Statement-II is correct
 (3) Both Statement-I and Statement-II are correct
 (4) Statement-I is correct but Statement-II is incorrect

Ans. (2)

Sol. $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$

$$\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$$

$$\vec{a} \times \vec{r} = \vec{a} \times \vec{b}; \vec{a} \cdot \vec{r} = 0$$

$$\Rightarrow \vec{a} \times (\vec{r} - \vec{b}) = \vec{0}$$

$$\Rightarrow \vec{a} = \lambda(\vec{r} - \vec{b})$$

$$\vec{a} \cdot \vec{a} = \lambda(\vec{a} \cdot \vec{r} - \vec{a} \cdot \vec{b})$$

$$14 = -7\lambda \Rightarrow \lambda = -2$$

$$\frac{-\vec{a}}{2} = \vec{r} - \vec{b} \Rightarrow \vec{r} = \vec{b} - \frac{\vec{a}}{2}$$

$$= \frac{2\vec{b} - \vec{a}}{2} = \frac{3\hat{i} + \hat{k}}{2}$$

Statement (I) is incorrect

$$\cos 2A + \cos 2B + \cos 2C \geq -\frac{3}{2}$$

$$2A + 2B + 2C = 2\pi$$

$$\cos 2A + \cos 2B + \cos 2C$$

$$= -1 - 4 \cos A \cdot \cos B \cdot \cos C$$

$$\geq -1 - 4 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= -\frac{3}{2}$$

Statement (II) is correct.



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11. $\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\int_{x^3}^{(\pi/2)^3} (\sin(2t^{1/3}) + \cos(t^{1/3})) dt}{\left(x - \frac{\pi}{2}\right)^2} \right)$ is equal

to :

- (1) $\frac{9\pi^2}{8}$ (2) $\frac{11\pi^2}{10}$
 (3) $\frac{3\pi^2}{2}$ (4) $\frac{5\pi^2}{9}$

Ans. (1)

Sol. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{0 - \{\sin(2x) + \cos(x)\} \cdot 3x^2}{2\left(x - \frac{\pi}{2}\right)}$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\{2 \sin x \cos x + \cos x\} 3x^2}{2\left(x - \frac{\pi}{2}\right)}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \left\{ \frac{2 \sin x \sin\left(\frac{\pi}{2} - x\right)}{2\left(x - \frac{\pi}{2}\right)} + \frac{\sin\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} \right\} 3x^2$$

$$= \left(1(1) + \frac{1}{2}\right) 3\left(\frac{\pi}{2}\right)^2$$

$$= \frac{9\pi^2}{8}$$

12. The sum of the coefficient of $x^{2/3}$ and $x^{-2/5}$ in the binomial expansion of $\left(x^{2/3} + \frac{1}{2}x^{-2/5}\right)^9$ is :

- (1) 21/4 (2) 69/16
 (3) 63/16 (4) 19/4

Ans. (1)

Sol. $T_{r+1} = {}^9C_r (x^{2/3})^{9-r} \left(\frac{x^{-2/5}}{2}\right)^r$

$$= {}^9C_r \left(\frac{1}{2}\right)^r (r)^{\left(\frac{6-2r}{3} - \frac{2r}{5}\right)}$$

for coefficient of $x^{2/3}$, put $6 - \frac{2r}{3} - \frac{2r}{5} = \frac{2}{3}$

$$\Rightarrow r = 5$$

\therefore Coefficient of $x^{2/3}$ is ${}^9C_5 \left(\frac{1}{5}\right)^5$

For coefficient of $x^{-2/5}$, put $6 - \frac{2r}{3} - \frac{2r}{5} = -\frac{2}{5}$

$$\Rightarrow r = 6$$

Coefficient of $x^{-2/5}$ is ${}^9C_6 \left(\frac{1}{2}\right)^6$

Sum = ${}^9C_5 \left(\frac{1}{2}\right)^5 + {}^9C_6 \left(\frac{1}{2}\right)^6 = \frac{21}{4}$

13. Let $B = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$ and A be a 2×2 matrix such that

$AB^{-1} = A^{-1}$. If $BCB^{-1} = A$ and $C^4 + \alpha C^2 + \beta I = O$, then $2\beta - \alpha$ is equal to :

- (1) 16 (2) 2
 (3) 8 (4) 10

Ans. (4)

Sol. $BCB^{-1} = A$

$$\Rightarrow (BCB^{-1})(BCB^{-1}) = A.A$$

$$\Rightarrow BCB^{-1} = A^2$$

$$\Rightarrow BC^2B^{-1} = A^2$$

$$\Rightarrow B^{-1}(BC^2B^{-1})B = B^{-1}(A.A)B$$

From equation (1)

$$C^2 = A^{-1}.A.B$$

$$C^2 = B$$

Also $AB^{-1} = A^{-1}$

$$\Rightarrow AB^{-1}.A = A^{-1}.A = I$$

$$\Rightarrow A^{-1}(AB^{-1}A) = A^{-1}.I$$

$$B^{-1}A = A^{-1}$$

Now characteristics equation of C^2 is

$$|C^2 - \lambda I| = 0$$

$$|B - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(5-\lambda) - 3 = 0 \Rightarrow (\lambda^2 - 6\lambda + 5) - 3 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 2 = 0$$

$$\Rightarrow \beta^2 - 6\beta + 2I = 0$$

$$\Rightarrow C^4 - 6C^2 + 2I = 0$$

$$\alpha = -6$$

$$\beta = 2$$

$$\therefore 2\beta - \alpha = 4 + 6 = 10$$

14. If $\log_e y = 3 \sin^{-1}x$, then $(1-x)^2 y'' - xy'$ at $x = \frac{1}{2}$

is equal to :

(1) $9e^{\pi/6}$

(2) $3e^{\pi/6}$

(3) $3e^{\pi/2}$

(4) $9e^{\pi/2}$

Ans. (4)

Sol. $\ln(y) = 3 \sin^{-1}x$

$$\frac{1}{y} \cdot y' = 3 \left(\frac{1}{\sqrt{1-x^2}} \right)$$

$$\Rightarrow y' = \frac{3y}{\sqrt{1-x^2}} \text{ at } x = \frac{1}{2}$$

$$\Rightarrow y' = \frac{3e^{3\left(\frac{\pi}{6}\right)}}{\sqrt{3}} = 2\sqrt{3}e^{\frac{\pi}{2}}$$

$$\Rightarrow y'' = 3 \left(\frac{\sqrt{1-x^2} y' - y \frac{1}{2\sqrt{1-x^2}} (-2x)}{(1-x^2)^2} \right)$$

$$\Rightarrow (1-x^2)y'' = 3 \left(3y + \frac{xy}{\sqrt{1-x^2}} \right)$$

$$\downarrow \text{ at } x = \frac{1}{2}, y = e^{3\sin^{-1}\left(\frac{1}{2}\right)} = e^{3\left(\frac{\pi}{6}\right)} = e^{\frac{\pi}{2}}$$

$$(1-x^2)y'' \Big|_{\text{at } x=\frac{1}{2}} = 3 \left(3e^{\frac{\pi}{2}} + \frac{\frac{1}{2} \left(e^{\frac{\pi}{2}} \right)}{\frac{\sqrt{3}}{2}} \right)$$

$$= 3e^{\frac{\pi}{2}} \left(3 + \frac{1}{\sqrt{3}} \right)$$

$$(1-x^2)y'' - xy' \Big|_{\text{at } x=\frac{1}{2}}$$

$$= 3e^{\frac{\pi}{2}} \left(3 + \frac{1}{\sqrt{3}} \right) - \frac{1}{2} \left(2\sqrt{3}e^{\frac{\pi}{2}} \right) = 9e^{\frac{\pi}{2}}$$

15. The integral $\int_{1/4}^{3/4} \cos \left(2 \cot^{-1} \sqrt{\frac{1-x}{1+x}} \right) dx$ is equal

to:

(1) $-1/2$

(2) $1/4$

(3) $1/2$

(4) $-1/4$

Ans. (4)

Sol. $I = \int_{1/4}^{3/4} \cos \left(2 \cot^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right) dx \right)$

$$\int_{1/4}^{3/4} \cos \left(2 \left(\tan^{-1} \sqrt{\frac{1+x}{1-x}} \right) \right) dx$$

$$\int_{1/4}^{3/4} \frac{1 - \tan^2 \left(\tan^{-1} \sqrt{\frac{1+x}{1-x}} \right)}{1 + \tan^2 \left(\tan^{-1} \sqrt{\frac{1+x}{1-x}} \right)} dx$$

$$= \int_{1/4}^{3/4} \frac{1 - \left(\frac{1+x}{1-x} \right)}{1 + \left(\frac{1+x}{1-x} \right)} dx = \int_{1/4}^{3/4} \frac{-2x}{2} dx$$

$$= \int_{1/4}^{3/4} (-x) dx = - \left(\frac{x^2}{2} \right)_{1/4}^{3/4}$$

$$= - \frac{1}{2} \left[\frac{9}{16} - \frac{1}{16} \right]$$

$$= - \frac{1}{4}$$



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16. Let a, ar, ar^2, \dots be an infinite G.P. If $\sum_{n=0}^{\infty} ar^n = 57$ and $\sum_{n=0}^{\infty} a^3 r^{3n} = 9747$, then $a + 18r$ is

equal to :

- (1) 27 (2) 46
(3) 38 (4) 31

Ans. (4)

Sol. $\sum_{n=0}^{\infty} ar^n = 57$

$a + ar + ar^2 + \dots = 57$

$\frac{a}{1-r} = 57 \dots\dots\dots (I)$

$\sum_{n=0}^{\infty} a^3 r^{3n} = 9747$

$a^3 + a^3 \cdot r^3 + a^3 \cdot r^6 + \dots = 9747$

$\frac{a^3}{1-r^3} = 9747 \dots\dots\dots (II)$

$\frac{(I)^3}{(II)} \Rightarrow \frac{(1-r)^3}{a^3} = \frac{57^3}{9717} = 19$

On solving, $r = \frac{2}{3}$ and $r = \frac{3}{2}$ (rejected)

$a = 19$

$\therefore a + 18r = 19 + 18 \times \frac{2}{3} = 31$

17. If an unbiased dice is rolled thrice, then the probability of getting a greater number in the i^{th} roll than the number obtained in the $(i-1)^{\text{th}}$ roll, $i = 2, 3$, is equal to :

- (1) 3/54 (2) 2/54
(3) 5/54 (4) 1/54

Ans. (3)

Sol. Favourable cases = 6C_3

Total outcomes = 6^3

Probability of getting greater number than previous

one = $\frac{{}^6C_3}{6^3} = \frac{20}{216} = \frac{5}{54}$

18. The value of the integral $\int_{-1}^2 \log_e (x + \sqrt{x^2 + 1}) dx$

is :

(1) $\sqrt{5} - \sqrt{2} + \log_e \left(\frac{9 + 4\sqrt{5}}{1 + \sqrt{2}} \right)$

(2) $\sqrt{2} - \sqrt{5} + \log_e \left(\frac{9 + 4\sqrt{5}}{1 + \sqrt{2}} \right)$

(3) $\sqrt{5} - \sqrt{2} + \log_e \left(\frac{7 + 4\sqrt{5}}{1 + \sqrt{2}} \right)$

(4) $\sqrt{2} - \sqrt{5} + \log_e \left(\frac{7 + 4\sqrt{5}}{1 + \sqrt{2}} \right)$

Ans. (2)

Sol. $I = \int_{-1}^2 1 \cdot \log_e (x + \sqrt{x^2 + 1}) dx$

$= x \log_e (x + \sqrt{x^2 + 1}) - \int_{-1}^2 \left(\frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} \right) dx$

$= x \log_e (x + \sqrt{x^2 + 1}) - \int_{-1}^2 \frac{x}{\sqrt{x^2 + 1}} dx$

$= x \log_e (x + \sqrt{x^2 + 1}) - \sqrt{x^2 + 1} \Big|_{-1}^2$

$= (2 \log_e (2 + \sqrt{5}) - \sqrt{5})$

$- (-\log_e (-1 + \sqrt{2}) - \sqrt{2})$

$= \log_e (2 + \sqrt{5})^2 - \sqrt{5} + \log_e (\sqrt{2} - 1) + \sqrt{2}$

$= \log_e (2 + \sqrt{5})^2 - \sqrt{5} + \log_e (\sqrt{2} - 1) + \sqrt{2}$

$$= \sqrt{2} - \sqrt{5} + \log_e \left(\frac{(2 + \sqrt{5})^2}{\sqrt{2+1}} \right)$$

$$= \sqrt{2} - \sqrt{5} + \log_e \left(\frac{9 + 4\sqrt{5}}{\sqrt{2+1}} \right)$$

19. Let $\alpha, \beta; \alpha > \beta$, be the roots of the equation

$$x^2 - \sqrt{2}x - \sqrt{3} = 0. \text{ Let } P_n = \alpha^n - \beta^n, n \in \mathbb{N}. \text{ Then}$$

$$(11\sqrt{3} - 10\sqrt{2}) P_{10} + (11\sqrt{2} + 10) P_{11} - 11P_{12} \text{ is}$$

equal to :

(1) $10\sqrt{2}P_9$

(2) $10\sqrt{3}P_9$

(3) $11\sqrt{2}P_9$

(4) $11\sqrt{3}P_9$

Ans. (2)

Sol. $x^2 - \sqrt{2}x - \sqrt{3} = 0 \left\langle \begin{matrix} \alpha \\ \beta \end{matrix} \right.$

$$\alpha^{n+2} - \sqrt{2}\alpha^{n+1} - \sqrt{3}\alpha^n = 0$$

$$\text{and } \beta^{n+2} - \sqrt{2}\beta^{n+1} - \sqrt{3}\beta^n = 0$$

Subtracting

$$(\alpha^{n+2} - \beta^{n+2}) - \sqrt{2}(\alpha^{n+1} - \beta^{n+1}) - \sqrt{3}(\alpha^n - \beta^n) = 0$$

$$\Rightarrow P_{n+2} - \sqrt{2}P_{n+1} - \sqrt{3}P_n = 0$$

Put $n = 10$

$$P_{12} - \sqrt{2}P_{11} - \sqrt{3}P_{10} = 0$$

$n = 9$

$$P_{11} - \sqrt{2}P_{10} - \sqrt{3}P_9 = 0$$

$$11(\sqrt{3}P_{10} + \sqrt{2}P_{11} - P_{11}) - 10(\sqrt{2}P_{10} - P_{11})$$

$$= 0 - 10(-\sqrt{3}P_9) = 10\sqrt{3}P_9$$

20. Let $\vec{a} = 2\hat{i} + \alpha\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$, $\vec{c} = \beta\hat{j} - \hat{k}$, where α and β are integers and $\alpha\beta = -6$. Let the values of the ordered pair (α, β) for which the area of the parallelogram of diagonals $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$ is $\frac{\sqrt{21}}{2}$, be (α_1, β_1) and (α_2, β_2) .

Then $\alpha_1^2 + \beta_1^2 - \alpha_2\beta_2$ is equal to

(1) 17 (2) 24

(3) 21 (4) 19

Ans. (4)

Sol. Area of parallelogram = $\frac{1}{2} |\vec{d}_1 \times \vec{d}_2|$

$$A = \frac{1}{2} |(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})| = \frac{\sqrt{21}}{2}$$

$$\text{so, } \vec{a} + \vec{b} = \hat{i} + \alpha\hat{j} + 2\hat{k}$$

$$\vec{b} + \vec{c} = -\hat{i} + \beta\hat{j}$$

$$(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \alpha & 2 \\ -1 & \beta & 0 \end{vmatrix}$$

$$= \hat{i}(-2\beta) - \hat{j}(2) + \hat{k}(\beta + \alpha)$$

$$|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})| = \sqrt{4\beta^2 + 4 + (\alpha + \beta)^2} = \sqrt{21}$$

$$4\beta^2 + 4 + \alpha^2 + \beta^2 + 2\alpha\beta = 21$$

$$\alpha^2 + 5\beta^2 - 12 = 17$$

$$\alpha^2 + 5\beta^2 = 29$$

$$\text{and } \alpha\beta = -6$$

and given α, β are integers

so,

$$\alpha = -3, \beta = 2$$

or

$$\alpha = 3, \beta = -2$$

$$(\alpha_1, \beta_1) = (-3, 2)$$

$$(\alpha_2, \beta_2) = (3, -2)$$

$$\alpha_1^2 + \beta_1^2 - \alpha_2\beta_2 = 9 + 4 + 6 = 19$$



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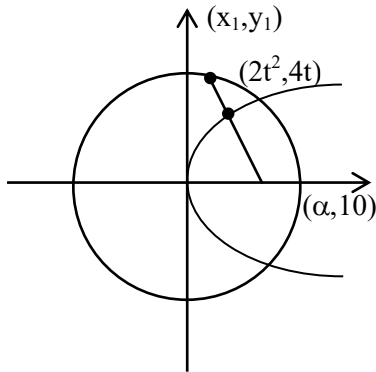
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SECTION-B

21. Consider the circle $C : x^2 + y^2 = 4$ and the parabola $P : y^2 = 8x$. If the set of all values of α , for which three chords of the circle C on three distinct lines passing through the point $(\alpha, 0)$ are bisected by the parabola P is the interval (p, q) , then $(2q - p)^2$ is equal to _____.

Ans. (80)

Sol.



$$T = S_1$$

$$xx_1 + yy_1 = x_1^2 + y_1^2$$

$$\alpha x_1 = x_1^2 + y_1^2$$

$$\alpha(2t^2) = 4t^4 + 16t^2$$

$$\alpha = 2t^2 + 8$$

$$\frac{\alpha - 8}{2} = t^2$$

$$\text{Also, } 4t^4 + 16t^2 - 4 < 0$$

$$t^2 = -2 + \sqrt{5}$$

$$\alpha = 4 + 2\sqrt{5}$$

$$\therefore \alpha \in (8, 4 + 2\sqrt{5})$$

$$\therefore (2q - p)^2 = 80$$

22. Let the set of all values of p , for which $f(x) = (p^2 - 6p + 8)(\sin^2 2x - \cos^2 2x) + 2(2 - p)x + 7$ does not have any critical point, be the interval (a, b) . Then $16ab$ is equal to _____.

Ans. (252)

Sol. $f(x) = -(p^2 - 6p + 8) \cos 4x + 2(2 - p)x + 7$

$$f'(x) = +4(p^2 - 6p + 8) \sin 4x + (4 - 2p) \neq 0$$

$$\sin 4x \neq \frac{2p - 4}{4(p - 4)(p - 2)}$$

$$\sin 4x \neq \frac{2(p - 2)}{4(p - 4)(p - 2)}$$

$$p \neq 2$$

$$\sin 4x \neq \frac{1}{2(p - 4)}$$

$$\Rightarrow \left| \frac{1}{2(p - 4)} \right| > 1$$

on solving we get

$$\therefore p \in \left(\frac{7}{2}, \frac{9}{2} \right)$$

$$\text{Hence } a = \frac{7}{2}, b = \frac{9}{2}$$

$$\therefore 16ab = 252$$

23. For a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, suppose $f'(x) = 3f(x) + \alpha$, where $\alpha \in \mathbb{R}$, $f(0) = 1$ and $\lim_{x \rightarrow -\infty} f(x) = 7$. Then $9f(-\log_e 3)$ is equal to _____.

Ans. (61)

Sol. $\frac{dy}{dx} - 3y = \alpha$

$$\text{If } y = e^{\int -3 dx} = e^{-3x}$$

$$\therefore y - e^{-3x} = \int e^{-3x} \cdot \alpha dx$$

$$y e^{-3x} = \frac{\alpha e^{-3x}}{-3} + c$$

$$(* e^{3x})$$

$$y = \frac{\alpha}{-3} + C \cdot e^{3x}$$

on substituting $x = 0, y = 1$

$$x \rightarrow -\infty, y = 7$$

$$\text{we get } y = 7 - 6e^{3x}$$

$$9f(-\log_e 3) = 61$$



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24. The number of integers, between 100 and 1000 having the sum of their digits equals to 14, is _____.

Ans. (70)

Sol. $N = a b c$

(i) All distinct digits

$$a + b + c = 14$$

$$a \geq 1$$

$$b, c \in \{0 \text{ to } 9\}$$

by hit & trial : 8 cases

$$(6, 5, 3) \quad (8, 6, 0) \quad (9, 4, 1)$$

$$(7, 6, 1) \quad (8, 5, 1) \quad (9, 3, 2)$$

$$(7, 5, 2) \quad (8, 4, 2)$$

$$(7, 4, 3) \quad (9, 5, 0)$$

(ii) 2 same, 1 diff $a = b ; c$

$$2a + c = 14$$

by values :

$$\left. \begin{array}{l} (3, 8) \\ (4, 6) \\ (5, 4) \\ (6, 2) \\ (7, 0) \end{array} \right\} \begin{array}{l} \text{Total} \\ 3! \\ 2! \end{array} \times 5 - 1$$

$$= 14 \text{ cases}$$

(iii) all same :

$$3a = 14$$

$$a = \frac{14}{3} \times \text{rejected}$$

$$0 \text{ cases}$$

Hence, Total cases :

$$8 \times 3! + 2 \times (4) + 14$$

$$= 48 + 22$$

$$= 70$$

25. Let $A = \{(x, y) : 2x + 3y = 23, x, y \in \mathbb{N}\}$ and $B = \{x : (x, y) \in A\}$. Then the number of one-one functions from A to B is equal to _____.

Ans. (24)

Sol. $2x + 3y = 23$

$$x = 1 \quad y = 7$$

$$x = 4 \quad y = 5$$

$$x = 7 \quad y = 3$$

$$x = 10 \quad y = 1$$

A B

$$(1, 7) \quad 1$$

$$(4, 5) \quad 4$$

$$(7, 3) \quad 7$$

$$(10, 1) \quad 10$$

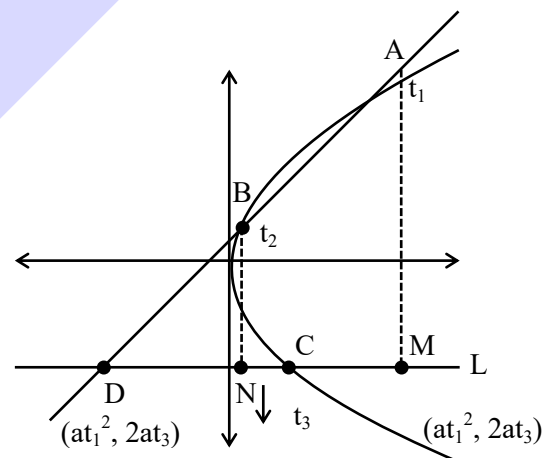
The number of one-one functions from A to B is equal to 4!

26. Let A, B and C be three points on the parabola $y^2 = 6x$ and let the line segment AB meet the line L through C parallel to the x-axis at the point D. Let M and N respectively be the feet of the perpendiculars from A and B on L.

Then $\left(\frac{AM \cdot BN}{CD}\right)^2$ is equal to _____.

Ans. (36)

Sol.



Sol.

$$m_{AB} = m_{AD}$$

$$\Rightarrow \frac{2}{t_1 + t_2} = \frac{2a(t_1 - t_3)}{at_1^2 - \alpha}$$

$$\Rightarrow at_1^2 - \alpha = a\{t_1^2 - t_1t_3 + t_1t_2 - t_2t_3\}$$

$$\Rightarrow \alpha = a(t_1t_3 + t_2t_3 - t_1t_2)$$

$$AM = |2a(t_1 - t_3)|, \quad BN = |2a(t_2 - t_3)|,$$

$$CD = |at_3^2 - \alpha|$$



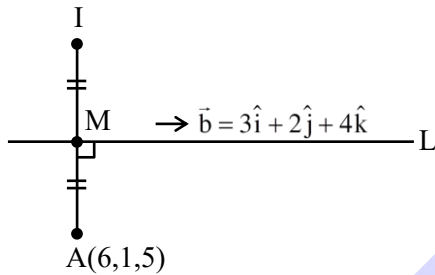
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$$\begin{aligned}
 CD &= |at_3^2 - a(t_1t_3 + t_2t_3 - t_1t_2)| \\
 &= a|t_3^2 - t_1t_3 - t_2t_3 + t_1t_2| \\
 &= a|t_3(t_3 - t_1) - t_2(t_3 - t_1)| \\
 CD &= a|(t_3 - t_2)(t_3 - t_1)| \\
 \left(\frac{AM \cdot BN}{CD}\right)^2 &= \left\{ \frac{2a(t_1 - t_3) \cdot 2a(t_2 - t_3)}{a(t_3 - t_2)(t_3 - t_1)} \right\}^2 \\
 16a^2 &= 16 \times \frac{9}{4} = 36
 \end{aligned}$$

27. The square of the distance of the image of the point (6, 1, 5) in the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z-2}{4}$, from the origin is _____.

Ans. (62)



Sol.

Let $M(3\lambda + 1, 2\lambda, 4\lambda + 2)$

$$\begin{aligned}
 \vec{AM} \cdot \vec{b} &= 0 \\
 \Rightarrow 9\lambda - 15 + 4\lambda - 2 + 16\lambda - 12 &= 0 \\
 \Rightarrow 29\lambda &= 29 \\
 \Rightarrow \lambda &= 1 \\
 M(4, 2, 6), I &= (2, 3, 7) \\
 \text{Required Distance} &= \sqrt{4+9+49} = \sqrt{62} \\
 \text{Ans. 62}
 \end{aligned}$$

28. If $\left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \dots + \frac{1}{\alpha+1012}\right) - \left(\frac{1}{2 \cdot 1} + \frac{1}{4 \cdot 3} + \frac{1}{6 \cdot 5} + \dots + \frac{1}{2024 \cdot 2023}\right) = \frac{1}{2024}$, then α is equal to-
 Ans. (1011)

Sol. $\left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \dots + \frac{1}{\alpha+2012}\right) - \left\{\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{2023} - \frac{1}{2024}\right)\right\} = \frac{1}{2024}$

$$\begin{aligned}
 \Rightarrow \left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \dots + \frac{1}{\alpha+2012}\right) - \left\{\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \dots + \frac{1}{2023} - \frac{1}{2024} - 2\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2022}\right)\right\} &= \frac{1}{2024} \\
 \Rightarrow \left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \dots + \frac{1}{\alpha+2012}\right) - \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2023}\right) + \frac{1}{2024} + \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{1011}\right) &= \frac{1}{2024} \\
 \Rightarrow \frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \dots + \frac{1}{\alpha+2012} &= \frac{1}{1012} + \frac{1}{1013} + \dots + \frac{1}{2023} \\
 \Rightarrow \alpha &= 1011
 \end{aligned}$$

29. Let the inverse trigonometric functions take principal values. The number of real solutions of the equation $2 \sin^{-1} x + 3 \cos^{-1} x = \frac{2\pi}{5}$, is _____.

Ans. (0)

Sol. $2 \sin^{-1} x + 3 \cos^{-1} x = \frac{2\pi}{5}$

$$\begin{aligned}
 \Rightarrow \pi + \cos^{-1} x &= \frac{2\pi}{5} \\
 \Rightarrow \cos^{-1} x &= \frac{-3\pi}{5} \\
 \text{Not possible} \\
 \text{Ans. 0}
 \end{aligned}$$

30. Consider the matrices : $A = \begin{bmatrix} 2 & -5 \\ 3 & m \end{bmatrix}$, $B = \begin{bmatrix} 20 \\ m \end{bmatrix}$

and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Let the set of all m , for which the

system of equations $AX = B$ has a negative solution (i.e., $x < 0$ and $y < 0$), be the interval (a, b) .

Then $8 \int_a^b |A| dm$ is equal to _____.

Ans. (450)

Sol. $A = \begin{pmatrix} 2 & -5 \\ 3 & m \end{pmatrix}$, $B = \begin{pmatrix} 20 \\ m \end{pmatrix}$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x - 5y = 20 \quad \dots(1)$$

$$3x + my = m \quad \dots(2)$$

$$\Rightarrow y = \frac{2m - 60}{2m + 15}$$

$$y < 0 \Rightarrow m \in \left(\frac{-15}{2}, 30 \right)$$

$$x = \frac{25m}{2m + 15}$$

$$x < 0 \Rightarrow m \in \left(\frac{-15}{2}, 0 \right)$$

$$\Rightarrow m \in \left(\frac{-15}{2}, 0 \right)$$

$$|A| = 2m + 15$$

Now,

$$8 \int_{\frac{-15}{2}}^0 (2m + 15) dm = 8 \left\{ m^2 + 15m \right\}_{\frac{-15}{2}}^0$$

$$\Rightarrow 8 \left\{ - \left(\frac{225}{4} - \frac{225}{2} \right) \right\}$$

$$= 8 \times \frac{225}{4} = 450$$



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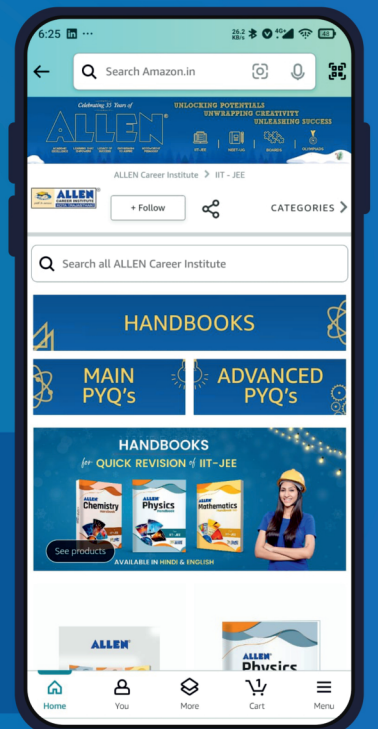
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