FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Saturday 06th April, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

1. If $f(x)=\begin{cases} x^3\sin\left(\frac{1}{x}\right) & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$, then

(1) $f''(0)=1$ (2) $f''\left(\frac{2}{\pi}\right)=\frac{24-\pi^2}{2\pi}$

(3) $f''\left(\frac{2}{\pi}\right)=\frac{12-\pi^2}{2\pi}$ (4) $f''(0)=0$

Ans. (2)

Sol. $f'(x) = 3x^2\sin\left(\frac{1}{x}\right)-x\cos\left(\frac{1}{x}\right)$

$$f''(x)=6x\sin\left(\frac{1}{x}\right)-3\cos\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)-\frac{\sin\left(\frac{1}{x}\right)}{x}$$

$$f''\left(\frac{2}{\pi}\right)=\frac{12}{\pi}-\frac{\pi}{2}=\frac{24-\pi^2}{2\pi}$$

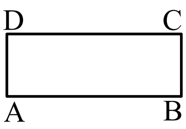
2. If A(3,1,–1), $B\left(\frac{5}{3},\frac{7}{3},\frac{1}{3}\right)$, C(2,2,1) and $D\left(\frac{10}{3},\frac{2}{3},\frac{-1}{3}\right)$ are the vertices of a quadrilateral ABCD, then its area is

(1) $\frac{4\sqrt{2}}{3}$ (2) $\frac{5\sqrt{2}}{3}$

(3) $2\sqrt{2}$ (4) $\frac{2\sqrt{2}}{3}$

Ans. (1)

Sol.



Area = $\frac{1}{2}\left|\overline{BD}\times\overline{AC}\right|$

$$\overline{BD}=\frac{5}{3}\hat{i}-\frac{5}{3}\hat{j}-\frac{2}{3}\hat{k}$$

$$\overline{AC}=\hat{i}-\hat{j}-2\hat{k}$$

3. $\displaystyle\int_0^{\pi/4}\frac{\cos^2x\sin^2x}{\left(\cos^3x+\sin^3x\right)^2}dx$ is equal to

(1) 1/12 (2) 1/9

(3) 1/6 (4) 1/3

Ans. (3)

Sol. Divide Nr & Dr by cosx

$$\int_0^{\pi/4}\frac{\tan^2x\sec^2x\,dx}{\left(1+\tan^3x\right)^2}dx$$

Let $1 + \tan^3x = t$

$\tan^2x\sec^2x\,dx = \frac{dt}{3}$

$$\frac{1}{3}\int_1^2\frac{dt}{t^2}=\frac{1}{6}$$

4. The mean and standard deviation of 20 observations are found to be 10 and 2, respectively. On respectively, it was found that an observation by mistake was taken 8 instead of 12. The correct standard deviation is

(1) $\sqrt{3.86}$ (2) 1.8

(3) $\sqrt{3.96}$ (4) 1.94

Ans. (3)

Sol. Mean $(\bar{x})=10$

$$\Rightarrow\frac{\Sigma x_i}{20}=10$$

$\Sigma x_i = 10\times20 = 200$

If 8 is replaced by 12, then $\Sigma x_i = 200 - 8 + 12 = 204$

$\therefore$ Correct mean $(\bar{x}) = \dfrac{\Sigma x_i}{20}$

$= \dfrac{204}{20} = 10.2$

$\because$ Standard deviation = 2

$\therefore$ Variance = $(\text{S.D.})^2 = 2^2 = 4$

$\Rightarrow \dfrac{\Sigma x_i^2}{20} - \left(\dfrac{\Sigma x_i}{20}\right)^2 = 4$

$\Rightarrow \dfrac{\Sigma x_i^2}{20} - (10)^2 = 4$

$\Rightarrow \dfrac{\Sigma x_i^2}{20} = 104$

$\Rightarrow \Sigma x_i^2 = 2080$

Now, replaced '8' observations by '12'

Then, $\Sigma x_i^2 = 2080 - 8^2 + 12^2 = 2160$

$\therefore$ Variance of removing observations

$\Rightarrow \dfrac{\Sigma x_i^2}{20} - \left(\dfrac{\Sigma x_i}{20}\right)^2$

$\Rightarrow \dfrac{2160}{20} - (10.2)^2$

$\Rightarrow 108 - 104.04$

$\Rightarrow 3.96$

Correct standard deviation

$= \sqrt{3.96}$

5. The function $f(x) = \dfrac{x^2 + 2x - 15}{x^2 - 4x + 9}$, $x \in R$ is

(1) both one-one and onto.

(2) onto but not one-one.

(3) neither one-one nor onto.

(4) one-one but not onto.

NTA Ans. (3)

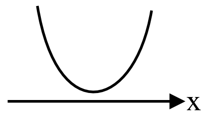
Ans. Bonus

Sol. $f(x) = \dfrac{(x+5)(x-3)}{x^2 - 4x + 9}$

Let $g(x) = x^2 - 4x + 9$

$D < 0$

$g(x) > 0$ for $x \in R$



$\therefore \left[\begin{array}{l} f(-5) = 0 \\ f(3) = 0 \end{array}\right.$

So, f(x) is many-one.

again,

$yx^2 - 4xy + 9y = x^2 + 2x - 15$

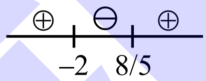
$x^2 (y - 1) - 2x(2y + 1) + (9y + 15) = 0$

for $\forall\ x \in R \Rightarrow D \geq 0$

$D = 4(2y + 1)^2 - 4(y - 1)(9y + 15) \geq 0$

$5y^2 + 2y + 16 \leq 0$

$(5y - 8)(y + 2) \leq 0$



$\boxed{y \in \left[-2, \dfrac{8}{5}\right]}$ range

Note : If function is defined from $f : R \to R$ then only correct answer is option (3)

$\Rightarrow$ Bonus

6. Let A = $\{n \in [100, 700] \cap N$: n is neither a multiple of 3 nor a multiple of 4$\}$. Then the number of elements in A is

(1) 300 (2) 280

(3) 310 (4) 290

Ans. (1)

Sol. $n(3) \Rightarrow$ multiple of 3

102, 105, 108, ... , 699

$T_n = 699 = 102 + (n - 1)(3)$

$n = 200$

$n(3) = 200$

$\because n(4) \Rightarrow$ multiple of 4

100, 104, 108, ..., 700

$T_n = 700 = 100 + (n-1)(4)$

$n = 151$

$n(4) = 151$

$n(3 \cap 4) \Rightarrow$ multiple of 3 & 4 both

108, 120, 132, ..., 696

$T_n = 696 = 108 + (n-1)(12)$

$n = 50$

$n(3 \cap 4) = 50$

$n(3 \cup 4) = n(3) + n(4) - n(3 \cap 4)$

$= 200 + 151 - 50$

$= 301$

$n(\overline{3 \cup 4})$ = Total – $n(3 \cup 4)$ = neither a multiple of 3 nor a multiple of 4

$= 601 - 301 = 300$

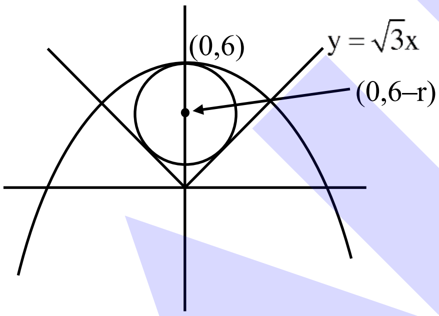
7. Let C be the circle of minimum area touching the parabola $y = 6 - x^2$ and the lines $y = \sqrt{3}|x|$. Then, which one of the following points lies on the circle C ?

(1) (2, 4) (2) (1, 2)

(3) (2, 2) (4) (1, 1)

Ans. (1)

Sol.



Equation of circle

$x^2 + (y-(6-r))^2 = r^2$

touches $\sqrt{3}x - y = 0$

$p = r$

$$\frac{|0-(6-r)|}{2} = r$$

$|r - 6| = 2r$

$r = 2$

$\therefore$ Circle $x^2 + (y-4)^2 = 4$

(2, 4) Satisfies this equation

8. For $\alpha, \beta \in R$ and a natural number n, let $A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2}+\alpha \\ 2r & 2 & n^2-\beta \\ 3r-2 & 3 & \frac{n(3n-1)}{2} \end{vmatrix}$. Then $2A_{10} - A_8$ is

(1) $4\alpha + 2\beta$ (2) $2\alpha + 4\beta$

(3) $2n$ (4) 0

Ans. (1)

Sol. $A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2}+\alpha \\ 2r & 2 & n^2-\beta \\ 3r-2 & 3 & \frac{n(3n-1)}{2} \end{vmatrix}$

$$2A_{10} - A_8 = \begin{vmatrix} 20 & 1 & \frac{n^2}{2}+\alpha \\ 40 & 2 & n^2-\beta \\ 56 & 3 & \frac{n(3n-1)}{2} \end{vmatrix} - \begin{vmatrix} 8 & 1 & \frac{n^2}{2}+\alpha \\ 16 & 2 & n^2-\beta \\ 22 & 3 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 12 & 1 & \frac{n^2}{2}+\alpha \\ 24 & 2 & n^2-\beta \\ 34 & 3 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \frac{n^2}{2}+\alpha \\ 0 & 2 & n^2-\beta \\ -2 & 3 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$\Rightarrow -2\left((n^2-\beta) - (n^2+2\alpha)\right)$

$\Rightarrow -2(-\beta - 2\alpha) \Rightarrow 4\alpha + 2\beta$

9. The shortest distance between the lines

$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$ and $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$ is

(1) $6\sqrt{3}$ (2) $4\sqrt{3}$

(3) $5\sqrt{3}$ (4) $8\sqrt{3}$

Ans. (2)

Sol. $\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$ & $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$

$$\text{S.D}=\frac{\left|(\bar{a}_2.\bar{a}_1).(\bar{b}_1.\bar{b}_2)\right|}{\left|\bar{b}_1\times\bar{b}_2\right|}$$

$a_1 = 3, -15, 9$ $b_1 = 2, -7, 5$

$a_2 = -1, 1, 9$ $b_2 = 2, 1, -3$

$a_2 - a_1 = -4, 16, 0$

$$\bar{b}_1\times\bar{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -7 & 5\\ 2 & 1 & -3\end{vmatrix}=\hat{i}(16)-\hat{j}(-16)+\hat{k}(16)$$

$16(\hat{i}+\hat{j}+\hat{k})$

$\left|\bar{b}_1\times\bar{b}_2\right|=16\sqrt{3}$

$\therefore\ (\bar{a}_2-\bar{a}_1).(\bar{b}_1-\bar{b}_2)=16[-4+16]=(16)(12)$

$$\text{S.D.}=\frac{(16)(12)}{16\sqrt{3}}=4\sqrt{3}$$

10. A company has two plants A and B to manufacture motorcycles. 60% motorcycles are manufactured at plant A and the remaining are manufactured at plant B. 80% of the motorcycles manufactured at plant A are rated of the standard quality, while 90% of the motorcycles manufactured at plant B are rated of the standard quality. A motorcycle picked up randomly from the total production is found to be of the standard quality. If p is the probability that it was manufactured at plant B, then 126p is

(1) 54 (2) 64

(3) 66 (4) 56

Ans. (1)

Sol.

	A	B
Manufactured	60%	40%
Standard quality	80%	90%

P(Manufactured at B / found standard quality) = ?

A : Found S.Q

B : Manufacture B

C : Manufacture A

$P(E_1)=\frac{40}{100}$

$P(E_2)=\frac{60}{100}$

$P(A/E_1)=\frac{90}{100}$

$P(A/E_2)=\frac{80}{100}$

$$\because P(E_1/A)=\frac{P(A/E_1)\,P(E_1)}{P(A/E_1)\,P(E_1)+P(A/E_2)\,P(E_2)}=\frac{3}{7}$$

$\therefore$ 126 P = 54

11. Let, α, β be the distinct roots of the equation

$x^2-(t^2-5t+6)x+1=0,\ t\in\mathbb{R}$ and $a_n=\alpha^n+\beta^n$.

Then the minimum value of $\frac{a_{2023}+a_{2025}}{a_{2024}}$ is

(1) 1/4 (2) –1/2

(3) –1/4 (4) 1/2

Ans. (3)

Sol. by newton's theorem

$a_{n+2}-(t^2-5t+6)a_{n+1}+a_n=0$

$\therefore\ a_{2025}+a_{2023}=(t^2-5t+6)\,a_{2024}$

$$\therefore\ \frac{a_{2025}+a_{2023}}{a_{2024}}=t^2-5t+6$$

$$\because\ t^2-5t+6=\left(t-\frac{5}{2}\right)^2-\frac{1}{4}$$

$$\therefore\ \text{minimum value}=-\frac{1}{4}$$

12. Let the relations R_1 and R_2 on the set $X = \{1, 2, 3, ..., 20\}$ be given by $R_1 = \{(x, y) : 2x - 3y = 2\}$ and $R_2 = \{(x, y) : -5x + 4y = 0\}$. If M and N be the minimum number of elements required to be added in R_1 and R_2, respectively, in order to make the relations symmetric, then M + N equals

(1) 8 (2) 16

(3) 12 (4) 10

Ans. (4)

Sol. $x = \{1, 2, 3, ...20\}$

$R_1 = \{(x, y) : 2x - 3y = 2\}$

$R_2 = \{(x, y) : -5x + 4y = 0\}$

$R_1 = \{(4, 2), (7, 4), (10, 6), (13, 8), (16, 10), (19, 12)\}$

$R_2 = \{(4, 5), (8, 10), (12, 15), (16, 20)\}$

in R_1 6 element needed

in R_2 4 element needed

So, total 6+4 = 10 element

13. Let a variable line of slope m > 0 passing through the point (4, –9) intersect the coordinate axes at the points A and B. the minimum value of the sum of the distances of A and B from the origin is

(1) 25 (2) 30

(3) 15 (4) 10

Ans. (1)

Sol. equation of line is

$y + 9 = m (x - 4)$

$\therefore\ A = \left(\frac{9+4m}{m}, 0\right)$

$B = (0, -9 - 4m)$

$\therefore\ OA + OB = \frac{9+4m}{m} + 9 + 4m$

$\because\ m > 0$

$= 13 + \frac{9}{m} + 4m$

$\because\ \frac{4m + \frac{9}{m}}{2} \geq \sqrt{36} \Rightarrow 4m + \frac{9}{m} \geq 12$

$\therefore\ OA + OB \geq 25$

14. The interval in which the function $f(x) = x^x$, $x > 0$, is strictly increasing is

(1) $\left(0, \frac{1}{e}\right]$ (2) $\left[\frac{1}{e^2}, 1\right)$

(3) $(0, \infty)$ (4) $\left[\frac{1}{e}, \infty\right)$

Ans. (4)

Sol. $f(x) = x^x ; x > 0$

$\ell ny = x\ell nx$

$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \ell nx$

$\frac{dy}{dx} = x^x(1 + \ell nx)$

for strictly increasing

$\frac{dy}{dx} \geq 0 \quad \Rightarrow x^x (1 + \ell nx) \geq 0$

$\Rightarrow \ell nx \geq -1$

$x \geq e^{-1}$

$x \geq \frac{1}{e}$

$x \in \left[\frac{1}{e}, \infty\right)$

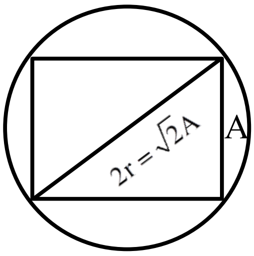
15. A circle in inscribed in an equilateral triangle of side of length 12. If the area and perimeter of any square inscribed in this circle are m and n, respectively, then $m + n^2$ is equal to

(1) 396 (2) 408

(3) 312 (4) 414

Ans. (2)

Sol. $\because r = \frac{\Delta}{s} = \frac{\sqrt{3}a^2}{4.\frac{3a}{2}} = \frac{a}{2\sqrt{3}} = \frac{12}{2\sqrt{3}} = 2\sqrt{3}$



$\therefore$ **A** $= r\sqrt{2} = 2\sqrt{6}$

Area $= m = A^2 = 24$

Perimeter $= n = 4A = 8\sqrt{6}$

$\therefore\ m + n^2 = 24 + 384$

$= 408$

16. The number of triangles whose vertices are at the vertices of a regular octagon but none of whose sides is a side of the octagon is

(1) 24 (2) 56

(3) 16 (4) 48

Ans. (3)

Sol. $\because$ no. of triangles having no side common with a n sided polygon $= \frac{{}^{n}C_1 \,.\, {}^{n-4}C_2}{3}$

$= \frac{{}^{8}C_1 \,.\, {}^{4}C_2}{3} = 16$

17. Let $y = y(x)$ be the solution of the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$, $y(1) = 0$. Then $y(0)$ is

(1) $\frac{1}{4}(e^{\pi/2} - 1)$ (2) $\frac{1}{2}(1 - e^{\pi/2})$

(3) $\frac{1}{4}(1 - e^{\pi/2})$ (4) $\frac{1}{2}(e^{\pi/2} - 1)$

Ans. (2)

Sol. $\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$

$\text{I.F.} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

$y.e^{\tan^{-1}x} = \int\left(\frac{e^{\tan^{-1}x}}{1+x^2}\right)e^{\tan^{-1}x}.dx$

Let $\tan^{-1}x = z$ $\qquad \therefore \frac{dx}{1+x^2} = dz$

$\therefore\ y.e^z = \int e^{2z}dz = \frac{e^{2z}}{2} + C$

$y.e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + C$

$\Rightarrow y = \frac{e^{\tan^{-1}x}}{2} + \frac{C}{e^{\tan^{-1}x}}$

$\because\ y(1) = 0 \ \Rightarrow 0 = \frac{e^{\pi/4}}{2} + \frac{C}{e^{\pi/4}} \quad \Rightarrow C = \frac{-e^{\pi/2}}{2}$

$\therefore\ y = \frac{e^{\tan^{-1}x}}{2} - \frac{e^{\pi/2}}{2e^{\tan^{-1}x}}$

$\boxed{\therefore y(0) = \frac{1-e^{\pi/2}}{2}}$

18. Let $y = y(x)$ be the solution of the differential equation $(2x\log_e x)\frac{dy}{dx} + 2y = \frac{3}{x}\log_e x$, $x > 0$ and $y(e^{-1}) = 0$. Then, $y(e)$ is equal to

(1) $-\frac{3}{2e}$ (2) $-\frac{2}{3e}$

(3) $-\frac{3}{e}$ (4) $-\frac{2}{e}$

Ans. (3)

Sol. $\frac{dy}{dx} + \frac{y}{x\,\ell n\,x} = \frac{3}{2x^2}$

$\therefore$ I.F. $= e^{\int \frac{1}{x\,\ell n\,x}dx} = e^{\ell n(\ell n(x))} = \ell nx$

$\therefore y\ell nx = \int \frac{3\ell n\,x}{2x^2}dx$

$= \frac{3\ell n\,x}{2}\int x^{-2}dx - \int\left(\frac{3}{2x}.\int x^{-2}\,dx\right)dx$

$= \frac{3\ell n\,x}{2}\left(-\frac{1}{x}\right) - \int \frac{3}{2x}\left(-\frac{1}{x}\right)dx$

$y.\ \ell nx = \frac{-3\ell nx}{2x} - \frac{3}{2x} + C$

$\because y(e^{-1}) = 0$

$\therefore 0\,(-1) = \frac{3e}{2} - \frac{3e}{2} + C \quad \Rightarrow C = 0$

$\therefore y = \frac{-3\ell nx}{2x} - \frac{3}{2x}$

$\boxed{\therefore y(e) = \frac{-3}{2e} - \frac{3}{2e} = \frac{-3}{e}}$

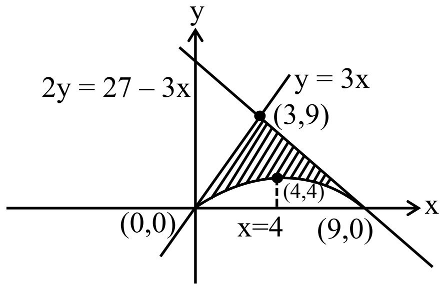
19. Let the area of the region enclosed by the curves $y = 3x$, $2y = 27 - 3x$ and $y = 3x - x\sqrt{x}$ be A. Then 10 A is equal to

(1) 184 (2) 154

(3) 172 (4) 162

Ans. (4)

Sol. $y = 3x$, $2y = 27 - 3x$ & $y = 3x - x\sqrt{x}$



$A = \int_0^3 3x - (3x - x\sqrt{x})dx + \int_3^9\left(\frac{27-3x}{2} - (3x - x\sqrt{x})\right)dx$

$A = \int_0^3 x^{3/2}dx + \int_3^9 \frac{27}{2} - \frac{9x}{2} + x^{3/2}dx$

$A = \left[\frac{2x^{5/2}}{5}\right]_0^3 + \frac{27}{2}[x]_3^9 - \frac{9}{2}\left[\frac{x^2}{2}\right]_3^9 + \left[\frac{2x^{5/2}}{5}\right]_3^9$

$A = \frac{2}{5}(3^{5/2}) + \frac{27}{2}(6) - \frac{9}{4}(72) + \frac{2}{5}(9^{5/2} - 3^{5/2})$

$A = \frac{2}{5}(3^{5/2}) + 81 - 162 + \frac{2}{5}\times 3^5 - \frac{2}{5}\times 3^{5/2}$

$A = \frac{486}{5} - 81 = \frac{81}{5}$

$10A = 162$

Ans. = 4

20. Let $f : (-\infty, \infty) - \{0\} \to R$ be a differentiable function such that $f'(1) = \lim_{a\to\infty} a^2 f\left(\frac{1}{a}\right)$.

Then $\lim_{a\to\infty} \frac{a(a+1)}{2}\tan^{-1}\left(\frac{1}{a}\right) + a^2 - 2\log_e a$ is equal to

(1) $\frac{3}{2} + \frac{\pi}{4}$ (2) $\frac{3}{8} + \frac{\pi}{4}$

(3) $\frac{5}{2} + \frac{\pi}{8}$ (4) $\frac{3}{4} + \frac{\pi}{8}$

Ans. (3)

Sol. $f:(-\infty,\infty)-\{0\}\rightarrow R$

$$f'(1)=\lim_{a\to\infty} a^2 f\left(\frac{1}{a}\right)$$

$$\lim_{a\to\infty}\frac{a(a+1)}{2}\tan^{-1}\left(\frac{1}{a}\right)+a^2-2\ell n(a)$$

$$\lim_{a\to\infty} a^2\left(\frac{\left(1+\frac{1}{a}\right)}{2}\tan^{-1}\left(\frac{1}{a}\right)+1-\frac{2}{a^2}\ell n(a)\right)$$

$$f(x)=\frac{1}{2}(1+x)\tan^{-1}(x)+1-2x^2\ \ell n(x)$$

$$f'(x)=\frac{1}{2}\left(\frac{1+x}{1+x^2}+\tan^{-1}(x)+4x\ell n(x)\right)+2x$$

$$f'(1)=\frac{1}{2}\left(1+\frac{\pi}{4}\right)+2$$

$$f'(1)=\frac{5}{2}+\frac{\pi}{8}$$

Ans. (3)

SECTION-B

21. Let $\alpha\beta\gamma=45$; $\alpha,\beta,\gamma\in R$. If $x(\alpha, 1, 2)+y(1,\beta,2)+z(2,3,\gamma)=(0,0,0)$ for some $x, y, z\in R$, $xyz\neq 0$, then $6\alpha+4\beta+\gamma$ is equal to________

Ans. (55)

Sol. $\alpha\beta\gamma=45$, $\alpha\beta\gamma\in R$

$x(\alpha,1,2)+y(1,\beta,2)+z(2,3,\gamma)=(0,0,0)$

$x, y, z\in R$, $xyz\neq 0$

$\alpha x+y+2z=0$

$x+\beta y+3z=0$

$2x+2y+\gamma z=0$

$xyz\neq 0\Rightarrow$ non-trivial

$$\begin{vmatrix}\alpha & 1 & 2\\ 1 & \beta & 3\\ 2 & 2 & \gamma\end{vmatrix}=0$$

$\Rightarrow \alpha(\beta\gamma-6)-1(\gamma-6)+2(2-2\beta)=0$

$\Rightarrow \alpha\beta\gamma-6\alpha-\gamma+6+4-4\beta=0$

$\Rightarrow 6\alpha+4\beta+\gamma=55$

22. Let a conic C pass through the point $(4, -2)$ and $P(x, y)$, $x\geq 3$, be any point on C. Let the slope of the line touching the conic C only at a single point P be half the slope of the line joining the points P and $(3, -5)$. If the focal distance of the point $(7, 1)$ on C is d, then 12d equals________.

Ans. (75)

Sol. $P(x, y)$ & $x\geq 3$

Slope of line at $P(x, y)$ will be $\frac{dy}{dx}=\frac{1}{2}\left(\frac{y+5}{x-3}\right)$

$$\Rightarrow 2\frac{dy}{(y+5)}=\frac{1}{(x-3)}dx$$

$\Rightarrow 2\ell n(y+5)=\ell n(x-3)+C$

Passes through $(4,-2)$

$\Rightarrow 2\ell n(3)=\ell n(1)+C$

$\Rightarrow C=2\ell n(3)$

$\Rightarrow 2\ell n(y+5)=\ell n(x-3)+2\ell n(3)$

$$\Rightarrow 2\left(\ell n\left(\frac{y+5}{3}\right)\right)=\ell n(x-3)$$

$$\Rightarrow \left(\frac{y+5}{3}\right)^2=(x-3)$$

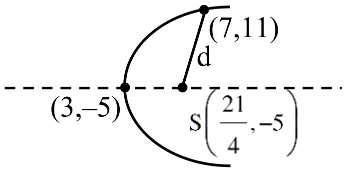
$\Rightarrow (y+5)^2=9(x-3)$

$\downarrow$

Parabola

$4a=9$

$$a=\frac{9}{4}$$



$$d=\sqrt{\left(\frac{7}{4}\right)^2+6^2}$$

$$d=\frac{\sqrt{625}}{4}$$

$$d=\frac{25}{4}$$

$12d = 75$

23. Let $r_k=\dfrac{\int_0^1\left(1-x^7\right)^k dx}{\int_0^1\left(1-x^7\right)^{k+1} dx}, k\in N$. Then the value of $\sum_{k=1}^{10}\dfrac{1}{7\left(r_k-1\right)}$ is equal to _______.

Ans. (65)

Sol. $I_K=\int 1.(1-x^7)^K dx$

$$I_K=(1-x^7)^K x\Big|_0^1+7K\int_0^1(1-x^7)^{K-1}x^6.xdx$$

$$I_K=-7K\int_0^1(1-x^7)^{K-1}((1-x^7)-1)dx$$

$I_K = -7K\, I_K + 7K\, I_{K-1}$

$$\Rightarrow \frac{I_K}{I_{K+1}}=\frac{7K+8}{7K+7}$$

$$r_K=\frac{7K+8}{7K+7}$$

$$r_K-1=\frac{1}{7(K+1)}$$

$$\Rightarrow 7(r_K-1)=\frac{1}{K+1}$$

$$\sum_{K=1}^{10}(K+1)=11(6)-1=65$$

24. Let x_1, x_2, x_3, x_4 be the solution of the equation $4x^4 + 8x^3 - 17x^2 - 12x + 9 = 0$ and

$$\left(4+x_1^2\right)\left(4+x_2^2\right)\left(4+x_3^2\right)\left(4+x_4^2\right)=\frac{125}{16}m.$$

Then the value of m is ________.

Ans. (221)

Sol. $4x^4 + 8x^3 - 17x^2 - 12x + 9$

$= 4(x - x_1)(x - x_2)(x - x_3)(x - x_4)$

Put $x = 2i$ & $-2i$

$64 - 64i + 68 - 24i + 9 = (2i - x_1)(2i - x_2)(2i - x_3)(2i - x_4)$

$= 141 - 88i$ ......(1)

$64 + 64i + 68 + 24i + 9 = 4(-2i - x_1)(-2i - x_2)(-2i - x_3)(-2i - x_4)$

$= 141 + 88i$ ....(2)

$$\frac{125}{16}m=\frac{141^2+88^2}{16}$$

$m = 221$

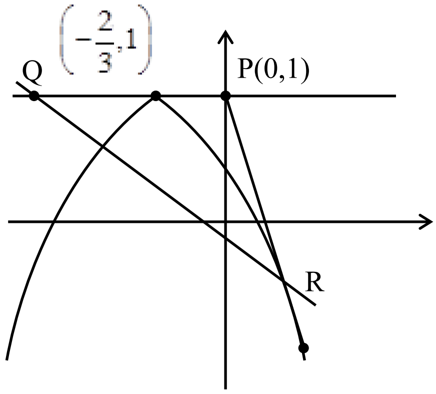
25. Let L_1, L_2 be the lines passing through the point P(0, 1) and touching the parabola $9x^2 + 12x + 18y - 14 = 0$. Let Q and R be the points on the lines L_1 and L_2 such that the ΔPQR is an isosceles triangle with base QR. If the slopes of the lines QR are m_1 and m_2. then $16\left(m_1^2+m_2^2\right)$ is equal to _______.

Ans. (68)

Sol. $9x^2 + 12x + 4 = -18(y - 1)$

$(3x + 2)^2 = -18(y - 1)$

$$\left(x+\frac{2}{3}\right)^2=-2(y-1)$$



(0, 1)

$y = mx + 1$

$\left(x+\frac{2}{3}\right)^2 = -2(y-1)$

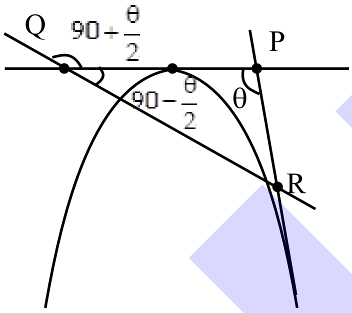
$(3x + 2)^2 = -18mx$

$9x^2 + (12 + 18m)x + 4 = 0$

$4(6 + 9m)^2 = 4(36)$

$6 + 9m = 6, -6$

$m = 0, \frac{-4}{3}$



$\tan\theta = -\frac{4}{3}$

$\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{-4}{3}$

$\left(\tan\frac{\theta}{2}-2\right)\left(2\tan\frac{\theta}{2}+1\right) = 0$

$\tan\frac{\theta}{2} = 2, \frac{-1}{2}$

$m_{QR} = \tan\left(90+\frac{\theta}{2}\right)$

$= -\cot\frac{\theta}{2}$

$m_1 = \frac{-1}{2}$ $\quad\quad$ $m_2 = \frac{-1}{-1/2} = 2$

$16\left(m_1^2+m_2^2\right) = 16\left(\frac{1}{4}+4\right)$

$= 4 + 64 = 68$

26. If the second, third and fourth terms in the expansion of $(x + y)^n$ are 135, 30 and $\frac{10}{3}$, respectively, then $6\left(n^3+x^2+y\right)$ is equal to ______.

Ans. (806)

Sol. ${}^nC_1x^{n-1}y = 135$ ....(i)

${}^nC_2x^{n-2}y^2 = 30$ ....(ii)

${}^nC_3x^{n-3}y^3 = \frac{10}{3}$ ....(iii)

By $\frac{(i)}{(ii)}$

$\frac{{}^nC_1}{{}^nC_2}\frac{x}{y} = \frac{9}{2}$ .....(iv)

By $\frac{(ii)}{(iii)}$

$\frac{{}^nC_2}{{}^nC_3}\frac{x}{y} = 9$ .....(v)

By $\frac{(iv)}{(v)}$

$\frac{{}^nC_1\,{}^nC_3}{{}^nC_2\,{}^nC_2} = \frac{1}{2}$

$\frac{2n^2(n-1)(n-2)}{6} = \frac{n(n-1)}{2}\frac{n(n-1)}{2}$

$4n - 8 = 3n - 3$

$\Rightarrow \boxed{n = 5}$

put in (v)

$$\frac{x}{y} = 9$$

$x = 9y$

put in (i)

$$^5C_1x^4\left(\frac{x}{9}\right) = 135$$

$x^5 = 27 \times 9$

$$\Rightarrow x = 3, \quad y = \frac{1}{3}$$

$$6\left(n^3 + x^2 + y\right)$$

$$= 6\left(125 + 9 + \frac{1}{3}\right)$$

$= 806$

27. Let the first term of a series be $T_1 = 6$ and its r^{th} term $T_r = 3\ T_{r-1} + 6^r$, $r = 2, 3, \ldots\ldots, n$. If the sum of the first n terms of this series is $\frac{1}{5}\left(n^2 - 12n + 39\right)$ $\left(4.6^n - 5.3^n + 1\right)$. Then n is equal to ______.

Ans. (6)

Sol. $T_r = 3T_{r-1} + 6^r$, $r = 2, 3, 4, \ldots n$

$T_2 = 3.T_1 + 6^2$

$T_2 = 3.6 + 6^2 \qquad \ldots(1)$

$T_3 = 3T_2 + 6^3$

$T_3 = 3T_2 + 6^3$

$T_3 = 3(3.6 + 6^2) + 6^3$

$T_3 = 3^2.6 + 3.6^2 + 6^3 \qquad \ldots(2)$

$T_r = 3^{r-1}.6 + 3^{r-2}.6^2 + \ldots + 6^r$

$$T_r = 3^{r-1}\cdot 6\left[1 + \frac{6}{3} + \left(\frac{6}{3}\right)^2 + \ldots + \left(\frac{6}{3}\right)^{r-1}\right]$$

$T_r = 3^{r-1}.6(1 + 2 + 2^2 + \ldots + 2^{r-1})$

$$T_r = 6\cdot 3^{r-1}1.\frac{(1-2^r)}{(-1)}$$

$T_r = 6.3^{r-1}.(2^r - 1)$

$$T_r = \frac{6\cdot 3^r}{3}.(2^r - 1)$$

$T_r = 2.(6^r - 3^r)$

$$S_n = 2\Sigma\left(6^r - 3^r\right)$$

$$S_n = 2.\left[\frac{6.(6^n - 1)}{5} - \frac{3.(3^n - 1)}{2}\right]$$

$$S_n = 2\left[\frac{12(6^n - 1) - 15(3^n - 1)}{10}\right]$$

$$S_n = \frac{3}{5}\left[4.6^4 - 5.3^n + 1\right]$$

$\therefore n^2 - 12n + 39 = 3$

$n^2 - 12n + 36 = 0$

$n = 6$

28. For $n \in N$, if $\cot^{-1}3 + \cot^{-1}4 + \cot^{-1}5 + \cot^{1} n = \frac{\pi}{4}$, then n is equal to ________.

Ans. (47)

Sol. $\cot^{-1}3 + \cot^{-1}4 + \cot^{-1}5 + \cot^{1} n = \frac{\pi}{4}$

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{n} = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{46}{48}\right) + \tan^{-1}\frac{1}{n} = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{23}{24}\right) + \tan^{-1}\frac{1}{n} = \frac{\pi}{4}$$

$$\tan^{-1}\frac{1}{n} = \tan^{-1}1 - \tan^{-1}\frac{23}{24}$$

$$\tan^{-1}\frac{1}{n} = \tan^{-1}\left(\frac{1 - \frac{23}{24}}{1 + \frac{23}{24}}\right)$$

$$\tan^{-1}\frac{1}{n} = \tan^{-1}\left(\frac{\frac{1}{24}}{\frac{47}{24}}\right)$$

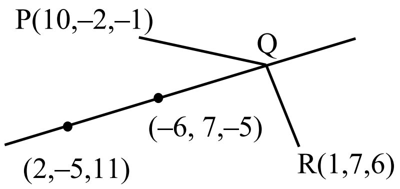
$$\tan^{-1}\frac{1}{n} = \tan^{-1}\frac{1}{47}$$

$n = 47$

29. Let P be the point (10, –2, –1) and Q be the foot of the perpendicular drawn from the point R(1, 7, 6) on the line passing through the points (2, –5, 11) and (–6, 7, –5). Then the length of the line segment PQ is equal to ________.

Ans. (13)

Sol.



Line : $\frac{x+6}{-8}=\frac{y-7}{12}=\frac{z+5}{-16}$

$\frac{x+6}{2}=\frac{y-7}{-3}=\frac{z+5}{4}=\lambda$

$Q(2\lambda-6,\ 7-3\lambda,\ 4\lambda-5)$

$\overline{QR}(2\lambda-7,-3\lambda,4\lambda-11)$

$\overline{QR}\cdot$ dr's of line = 0

$4\lambda - 14 + 9\lambda + 16\lambda - 44 = 0$

$29\lambda = 58 \Rightarrow \lambda = 2$

$Q(-2, 1, 3)$

$PQ=\sqrt{144+9+16}=\sqrt{169}=13$

30. Let $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=3\hat{i}+4\hat{j}-5\hat{k}$, and a vector $\vec{c}$ be such that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times\vec{c}=\hat{i}+8\hat{j}+13\hat{k}$. If $\vec{a}\cdot\vec{c}=13$, then $(24-\vec{b}\cdot\vec{c})$ is equal to ______.

Ans. (46)

Sol. $\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}=(1,8,13)$

$\vec{a}\times(\vec{a}\times\vec{b})+\vec{a}\times(\vec{a}\times\vec{c})+\vec{a}\times(\vec{b}\times\vec{c})$

$=\vec{a}\times(\hat{i}+8\hat{j}+13\hat{k})$

$(\vec{a}\cdot\vec{b})\vec{a}-a^2\vec{b}+(\vec{a}\cdot\vec{c})\vec{a}-a^2\vec{c}+(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\vec{a}\times(\hat{i}+8\hat{j}+13\hat{k})$

$\Rightarrow -26\vec{a}-29\vec{b}+13\vec{a}-29\vec{c}+13\vec{b}+26\vec{c}=\vec{a}\times(\hat{i}+8\hat{j}+13\hat{k})$

$\Rightarrow\quad -13\vec{a}-16\vec{b}-3\vec{c}=\vec{a}\times(\hat{i}+8\hat{j}+13\hat{k})$

$\Rightarrow\quad -13\vec{a}\cdot\vec{b}-16b^2-3\vec{b}\cdot\vec{c}=\left\{\vec{a}\times(\hat{i}+8\hat{j}+13\hat{k})\right\}\cdot\vec{b}$

$\Rightarrow\quad (-13)(-26)-16(50)-3\vec{b}\cdot\vec{c}=\begin{vmatrix}2 & -3 & 4\\ 1 & 8 & 13\\ 3 & 4 & -5\end{vmatrix}$

$\Rightarrow\quad -462-3\vec{b}\cdot\vec{c}=-396$

$\Rightarrow\quad \vec{b}\cdot\vec{c}=-22$

Hence $24-\vec{b}\cdot\vec{c}=46$