FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Friday 05th April, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

1. Let d be the distance of the point of intersection of the lines $\frac{x+6}{3}=\frac{y}{2}=\frac{z+1}{1}$ and $\frac{x-7}{4}=\frac{y-9}{3}=\frac{z-4}{2}$ from the point (7, 8, 9). Then $d^2 + 6$ is equal to :

(1) 72 (2) 69

(3) 75 (4) 78

Ans. (3)

Sol. $\frac{x+6}{3}=\frac{y}{2}=\frac{z+1}{1}=\lambda$ …(1)

$x = 3\lambda - 6, y = 2\lambda, z = \lambda - 1$

$\frac{x-7}{4}=\frac{y-9}{3}=\frac{z-4}{2}=\mu$ …(2)

$x = 4\mu + 7, y = 3\mu + 9, z = 2\mu + 4$

$3\lambda - 6 = 4\mu + 7 \Rightarrow 3\lambda - 4\mu = 13$ …(3) × 2

$2\lambda = 3\mu + 9 \Rightarrow 2\lambda - 3\mu = 9$ …(4) × 3

$6\lambda - 8\mu = 26$

$6\lambda - 9\mu = 27$

$-\quad + \quad -$

$\mu = -1$

$\Rightarrow 3\lambda - 4(-1) = 13$

$3\lambda = 9$

$\lambda = 3$

int. point (3, 6, 2) ; (7, 8, 9)

$d^2 = 16 + 4 + 49 = 69$

Ans. $d^2 + 6 = 69 + 6 = 75$

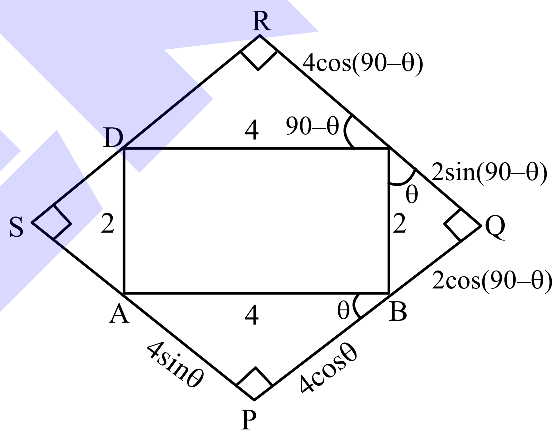
2. Let a rectangle ABCD of sides 2 and 4 be inscribed in another rectangle PQRS such that the vertices of the rectangle ABCD lie on the sides of the rectangle PQRS. Let a and b be the sides of the rectangle PQRS when its area is maximum. Then $(a + b)^2$ is equal to :

(1) 72 (2) 60

(3) 80 (4) 64

Ans. (1)

Sol.



Area = $(4\cos\theta + 2\sin\theta)(2\cos\theta + 4\sin\theta)$

$= 8\cos^2\theta + 16\sin\theta\cos\theta + 4\sin\theta\cos\theta + 8\sin^2\theta$

$= 8 + 20\sin\theta\cos\theta$

$= 8 + 10\sin2\theta$

Max Area = 8 + 10 = 18 ($\sin2\theta = 1$) $\theta = 45°$

$(a + b)^2 = (4\cos\theta + 2\sin\theta + 2\cos\theta + 4\sin\theta)^2$

$= (6\cos\theta + 6\sin\theta)^2$

$= 36(\sin\theta + \cos\theta)^2$

$= 36(\sqrt{2})^2$

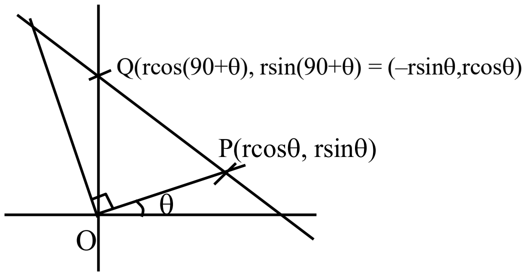
$= 72$

3. Let two straight lines drawn from the origin O intersect the line 3x + 4y = 12 at the points P and Q such that ΔOPQ is an isosceles triangle and ∠POQ = 90°. If $l = OP^2 + PQ^2 + QO^2$, then the greatest integer less than or equal to l is :

(1) 44 (2) 48
(3) 46 (4) 42

Ans. (3)

Sol.



3x + 4y = 12

3(rcosθ) + 4(rsinθ) = 12

r(3cosθ + 4sinθ) = 12 ...(1)

3(–rsinθ) + 4(rcosθ) = 12

r(–3sinθ + 4cosθ) = 12 ...(2)

$$\left(\frac{12}{r}\right)^2 + \left(\frac{12}{r}\right)^2 = (3\cos\theta + 4\sin\theta)^2 + (-3\sin\theta + 4\cos\theta)^2$$

$$2\left(\frac{12}{r}\right)^2 = 9 + 16$$

$$\frac{2\times144}{r^2} = 25 \Rightarrow 288 = 25r^2$$

$$\Rightarrow \frac{288}{25} = r^2$$

$$\Rightarrow \sqrt{2}\left(\frac{12}{5}\right) = r$$

$$\ell = OP^2 + PQ^2 + QO^2$$

$$\ell = r^2 + r^2 \quad + r^2(\cos\theta + \sin\theta)^2 + r^2(\sin\theta + \cos\theta)^2$$

$$= 2r^2 + r^2(1 + \sin2\theta + 1 - 2\sin2\theta)$$

$$= 2r^2 + 2r^2$$

$$= 4r^2$$

$$= 4\left(\frac{288}{25}\right) = \frac{1152}{25} = 46.08$$

$$[\ell] = 46$$

4. If y = y(x) is the solution of the differential equation $\frac{dy}{dx} + 2y = \sin(2x)$, $y(0) = \frac{3}{4}$, then $y\left(\frac{\pi}{8}\right)$ is equal to :

(1) $e^{-\pi/8}$ (2) $e^{-\pi/4}$
(3) $e^{\pi/4}$ (4) $e^{\pi/8}$

Ans. (2)

Sol. $\frac{dy}{dx} + 2y = \sin 2x$, $y(0) = \frac{3}{4}$

$$I.F = e^{\int 2dx} = e^{2x}$$

$$y.e^{2x} = \int e^{2x}\sin 2x\,dx$$

$$y.e^{2x} = \frac{e^{2x}(2\sin 2x - 2\cos 2x)}{4+4} + C$$

$$x = 0,\ y = \frac{3}{4} \Rightarrow \frac{3}{4}.1 = \frac{1(0-2)}{8} + C$$

$$\frac{3}{4} = -\frac{1}{4} + C$$

$$1 = C$$

$$y = \frac{2\sin 2x - 2\cos 2x}{8} + 1.e^{-2x}$$

$$x = \frac{\pi}{8},\quad y = \frac{1}{8}\left(2\sin\frac{\pi}{4} - 2\cos\frac{\pi}{4}\right) + e^{-2\left(\frac{\pi}{8}\right)}$$

$$y = 0 + e^{-\frac{\pi}{4}}$$

5. For the function

$f(x) = \sin x + 3x - \frac{2}{\pi}(x^2 + x)$, where $x \in \left[0, \frac{\pi}{2}\right]$, consider the following two statements :

(I) f is increasing in $\left(0, \frac{\pi}{2}\right)$.

(II) f′ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

Between the above two statements,

(1) only (I) is true.
(2) only (II) is true.
(3) neither (I) nor (II) is true.
(4) both (I) and (II) are true.

Ans. (4)

Sol. $f(x) = \sin x + 3x - \frac{2}{\pi}(x^2 + x) \quad x \in \left[0, \frac{\pi}{2}\right]$

$f'(x) = \cos x + 3 - \frac{2}{\pi}(2x + 1) > 0 \quad f(x) \uparrow$

$f'(x) = -\sin x + 0 - \frac{\pi}{2}(2)$

$= -\sin x - \frac{4}{\pi} < 0 \quad f'(x) \downarrow$

$0 < x < \frac{\pi}{2}$

$\Rightarrow -\frac{2}{\pi}\left(\underset{+1}{0} < \underset{+1}{2x} < \underset{+1}{\pi}\right)$

$\underset{+3}{-\frac{2}{\pi}} > \underset{+3}{\frac{-2}{\pi}(2x+1)} > \underset{+3}{-\frac{2}{\pi}(\pi+1)}$

$\underset{(+ve)}{3 - \frac{2}{\pi}} > 3 - \frac{2}{\pi}(2x+1) > \underset{(+ve)}{3 - \frac{2}{\pi}(\pi+1)}$

6. If the system of equations

$11x + y + \lambda z = -5$

$2x + 3y + 5z = 3$

$8x - 19y - 39z = \mu$

has infinitely many solutions, then $\lambda^4 - \mu$ is equal to :

(1) 49 (2) 45

(3) 47 (4) 51

Ans. (3)

Sol. $11x + y + \lambda z = -5$

$2x + 3y + 5z = 3$

$8x - 19y - 39z = \mu$

for infinite sol.

$$D = \begin{vmatrix} 11 & 1 & \lambda \\ 2 & 3 & 5 \\ 8 & -19 & -39 \end{vmatrix} = 0$$

$\Rightarrow 11(-117 + 95) - 1(-78 - 40) + \lambda(-38 - 24)$

$\Rightarrow 11(-22) + 118 - \lambda(62) = 0$

$\Rightarrow 62\lambda = 118 - 242$

$\Rightarrow \lambda = \frac{-124}{62} = -2$

$$D_1 = \begin{vmatrix} -5 & 1 & -2 \\ 3 & 3 & 5 \\ \mu & -19 & -39 \end{vmatrix} = 0$$

$\Rightarrow -5(-117 + 95) - 1(-117 - 5\mu) - 2(-57 - 3\mu) = 0$

$\Rightarrow -5(-22) + 117 + 5\mu + 114 + 6\mu = 0$

$\Rightarrow 11\mu = -110 - 231 = -341$

$\Rightarrow \mu = -31$

$\lambda^4 - \mu = (-2)^4 - (-31) = 16 + 31 = 47$

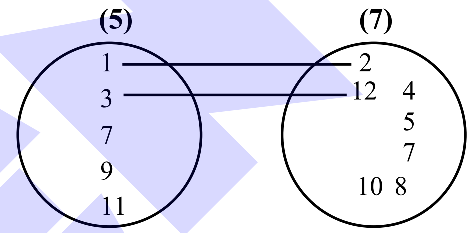
7. Let A = {1, 3, 7, 9, 11} and B = {2, 4, 5, 7, 8, 10, 12}. Then the total number of one-one maps $f : A \to B$, such that $f(1) + f(3) = 14$, is :

(1) 180 (2) 120

(3) 480 (4) 240

Ans. (4)

Sol.



A = {1, 3, 7, 9, 11}

B = {2, 4, 5, 7, 8, 10, 12}

$f(1) + f(3) = 14$

(i) 2 + 12

(ii) 4 + 10

$2 \times (2 \times 5 \times 4 \times 3) = 240$

8. If the function $f(x) = \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3}$, $x \in R$, is continuous at x = 0, then f(0) is equal to :

(1) 2 (2) –2

(3) 4 (4) –4

Ans. (4)

Sol. $f(x) = \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3}$

is continuous at x = 0

$$\lim_{x \to 0} = \frac{3x - \frac{(3x)^3}{\lfloor 3} + \ldots + \alpha\left(x - \frac{x^3}{\lfloor 3}\ldots\right) - \beta\left(1 - \frac{(3x)^2}{\lfloor 2}\ldots\right)}{x^3} = f(0)$$

$$\lim_{x\to 0} \frac{-\beta + x(3+\alpha) + \frac{9\beta x^2}{\lfloor 2} + \left(\frac{-27}{\lfloor 3} - \frac{\alpha}{\lfloor 3}\right)x^3 \ldots}{x^3} = f(0)$$

for exist

$$\beta = 0,\ 3+\alpha = 0,\ -\frac{27}{\lfloor 3} - \frac{\alpha}{\lfloor 3} = f(0)$$

$$\alpha = -3,\ -\frac{27}{6} - \frac{(-3)}{6} = f(0)$$

$$f(0) = \frac{-27+3}{6} = -4$$

9. The integral $\int_0^{\frac{\pi}{4}} \frac{136\sin x}{3\sin x + 5\cos x} dx$ is equal to :

(1) $3\pi - 50 \log_e 2 + 20 \log_e 5$

(2) $3\pi - 25 \log_e 2 + 10 \log_e 5$

(3) $3\pi - 10 \log_e\left(2\sqrt{2}\right) + 10 \log_e 5$

(4) $3\pi - 30 \log_e 2 + 20 \log_e 5$

Ans. (1)

Sol. $I = \int_0^{\pi/4} \frac{136\sin x}{3\sin x + 5\cos x} dx$

$136\sin x = A(3\sin x + 5\cos x) + B(3\cos x - 5\sin x)$

$136 = 3A - 5B$ …(1)

$0 = 5A + 3B$ …(2)

$$3B = -5A \Rightarrow B = -\frac{5}{3}A$$

$$136 = 3A - 5\left(-\frac{5}{3}A\right)$$

$$136 = 3A + \frac{25}{3}A$$

$$136 = \frac{34A}{3}$$

$$\Rightarrow A = \frac{136 \times 3}{34} = 12$$

$$B = \frac{-5}{3}(12) = -20$$

$$I = \int_0^{\pi/4} \frac{A(3\sin x + 5\cos x)}{3\sin x + 5\cos x} + \int_0^{\pi/4} \frac{B(3\cos x - 5\sin x)}{3\sin x + 5\cos x}$$

$$= A(x)_0^{\pi/4} + B\left[\ell n(3\sin x + 5\cos x)\right]_0^{\pi/4}$$

$$= 12\left(\frac{\pi}{4}\right) - 20\ell n\left(\frac{3}{\sqrt{2}} + \frac{5}{\sqrt{2}}\right) - \ell n(0+5)$$

$$= 3\pi - 20\ell n 4\sqrt{2} + 20\ell n 5$$

$$= 3\pi - 20 \times \frac{5}{2}\ell n 2 + 20\ell n 5$$

$$= 3\pi - 50\ell n 2 + 20\ell n 5$$

10. The coefficients a, b, c in the quadratic equation $ax^2 + bx + c = 0$ are chosen from the set {1, 2, 3, 4, 5, 6, 7, 8}. The probability of this equation having repeated roots is :

(1) $\frac{3}{256}$ (2) $\frac{1}{128}$

(3) $\frac{1}{64}$ (4) $\frac{3}{128}$

Ans. (3)

Sol. $ax^2 + bx + c = 0$

$a, b, c \in \{1, 2, 3, 4, 5, 6, 7, 8\}$

Repeated roots D = 0

$\Rightarrow b^2 - 4ac = 0 \Rightarrow b^2 = 4ac$

$$\text{Prob} = \frac{8}{8\times 8\times 8} = \frac{1}{64}$$

$\Rightarrow$ (a, b, c)

(1, 2, 1) ; (2, 4, 2) ; (1, 4, 4) ; (4, 4, 1) ; (3, 6, 3) ;

(2, 8, 8) ; (8, 8, 2) ; (4, 8, 4)

8 case

11. Let A and B be two square matrices of order 3 such that $|A| = 3$ and $|B| = 2$.

Then $|A^T A(\text{adj}(2A))^{-1} (\text{adj}(4B))(\text{adj}(AB))^{-1}AA^T|$ is equal to :

(1) 64 (2) 81

(3) 32 (4) 108

Ans. (1)

Sol. $|A| = 3, |B| = 2$

$|A^TA(\text{adj}(2A))^{-1} (\text{adj}(4B)) (\text{adj}(AB))^{-1}AA^T|$

$= 3\times3\times|(\text{adj}(2A))^{-1}| \times |\text{adj}(4B)| \times |(\text{adj}(AB))^{-1}|\times3\times3$

$\downarrow \qquad \downarrow \qquad \downarrow$

$\frac{1}{|\text{adj}(2A)|} \qquad 2^{12}\times 2^2 \qquad \frac{1}{|\text{adj}(AB)|}$

$= \frac{1}{2^6|\text{adj}A|} \qquad = \frac{1}{|\text{adj}B\cdot\text{adj}A|}$

$= \frac{1}{2^6\cdot3^2} \qquad = \frac{1}{2^2\cdot3^2}$

$$= 3^4\cdot\frac{1}{2^6\cdot3^2}\cdot2^{12}\cdot2^2\cdot\frac{1}{2^2\cdot3^2} = 64$$

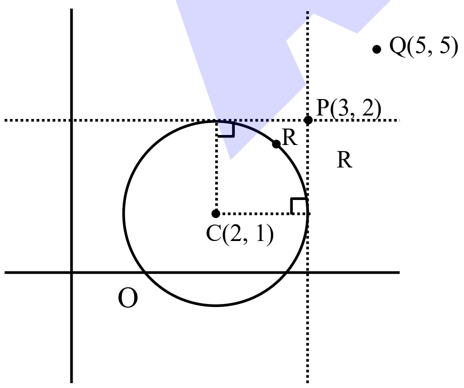
12. Let a circle C of radius 1 and closer to the origin be such that the lines passing through the point (3, 2) and parallel to the coordinate axes touch it. Then the shortest distance of the circle C from the point (5, 5) is :

(1) $2\sqrt{2}$ (2) 5

(3) $4\sqrt{2}$ (4) 4

Ans. (4)

Sol.



Coordinates of the centre will be (2, 1)

Equation of circle will be

$(x - 2)^2 + (y - 1)^2 = 1$

$QC = \sqrt{(5-2)^2 + (5-1)^2}$

$QC = 5$

shortest distance

$= RQ = CQ - CR$

$= 5 - 1$

$= 4$

13. Let the line $2x + 3y - k = 0$, $k > 0$, intersect the x-axis and y-axis at the points A and B, respectively. If the equation of the circle having the line segment AB as a diameter is $x^2 + y^2 - 3x - 2y = 0$ and the length of the latus rectum of the ellipse $x^2 + 9y^2 = k^2$ is $\frac{m}{n}$, where m and n are coprime, then $2m + n$ is equal to

(1) 10 (2) 11

(3) 13 (4) 12

Ans. (2)

Sol. Centre of the circle $= \left(\frac{3}{2}, 1\right)$

Equation of diameter $= 2x + 3y - k = 0$

$2\left(\frac{3}{2}\right) + 3(1) - k = 0$

$\Rightarrow k = 6$

Now, Equation of ellipse becomes

$x^2 + 9y^2 = 36$

$\frac{x^2}{6^2} + \frac{y^2}{2^2} = 1$

length of LR $= \frac{2b^2}{a} = \frac{2.2^2}{6} = \frac{8}{6} = \frac{4}{3} = \frac{m}{n}$

$\therefore 2m + n = 2(4) + 3 = 11$

14. Consider the following two statements :

Statement I : For any two non-zero complex numbers z_1, z_2

$(|z_1|+|z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \le 2(|z_1|+|z_2|)$ and

Statement II : If x, y, z are three distinct complex numbers and a, b, c are three positive real numbers such that $\frac{a}{|y-z|} = \frac{b}{|z-x|} = \frac{c}{|x-y|}$, then

$\frac{a^2}{y-z} + \frac{b^2}{z-x} + \frac{c^2}{x-y} = 1$.

Between the above two statements,

(1) both Statement I and Statement II are incorrect.

(2) Statement I is incorrect but Statement II is correct.

(3) Statement I is correct but Statement II is incorrect.

(4) both Statement I and Statement II are correct.

Ans. (3)

Sol. **Statement I :**

$(|z_1|+|z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|$

Since $\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \le \left|\frac{z_1}{|z_1|}\right| + \left|\frac{z_2}{|z_2|}\right|$

$\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \le \frac{|z_1|}{|z_1|} + \frac{|z_2|}{|z_2|}$

$\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \le 2$

$(|z_1|+|z_2|)\left(\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|\right) \le 2(|z_1|+|z_2|)$

$\therefore$ statement I is correct

For **Statement II :**

$\frac{a}{|y-z|} = \frac{b}{|z-x|} = \frac{c}{|x-y|}$

$\frac{a^2}{|y-z|^2} = \frac{b^2}{|z-x|^2} = \frac{c^2}{|x-y|^2} = \lambda$

$a^2 = \lambda(|y-z|^2) = \lambda(y-z)(\bar{y}-\bar{z})$

$b^2 = \lambda(z-x)(\bar{z}-\bar{x})$ and $c^2 = \lambda(x-y)(\bar{x}-\bar{y})$

$\frac{a^2}{y-z} + \frac{b^2}{z-x} + \frac{c^2}{x-y} = \lambda(\bar{y}-\bar{z}+\bar{z}-\bar{x}+\bar{x}-\bar{y}) = 0$

Statement II is false

15. Suppose $\theta \in \left[0, \frac{\pi}{4}\right]$ is a solution of $4\cos\theta - 3\sin\theta = 1$.

Then $\cos\theta$ is equal to :

(1) $\frac{4}{(3\sqrt{6}-2)}$ (2) $\frac{6-\sqrt{6}}{(3\sqrt{6}-2)}$

(3) $\frac{6+\sqrt{6}}{(3\sqrt{6}+2)}$ (4) $\frac{4}{(3\sqrt{6}+2)}$

Ans. (1)

Sol. $4\left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right) - 3\left(\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) = 1$

let $\tan\frac{\theta}{2} = t$

$\frac{4-4t^2-6t}{1+t^2} = 1$

$4 - 4t^2 - 6t = 1 + t^2$

$\Rightarrow 5t^2 + 6t - 3 = 0$

$\Rightarrow t = \dfrac{-6 \pm \sqrt{36 - 4(5)(-3)}}{2(5)}$

$= \dfrac{-6 \pm \sqrt{96}}{10}$

$= \dfrac{-6 \pm 4\sqrt{6}}{10}$

$t = \dfrac{-3 + 2\sqrt{6}}{5}$

$$\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1 - \left(\frac{2\sqrt{6}-3}{5}\right)^2}{1 + \left(\frac{2\sqrt{6}-3}{5}\right)^2} = \frac{1 - \left(\frac{24+9-12\sqrt{6}}{25}\right)}{1 + \left(\frac{24+9-12\sqrt{6}}{25}\right)}$$

$$= \frac{25-33+12\sqrt{6}}{25+33-12\sqrt{6}} = \frac{12\sqrt{6}-8}{58-12\sqrt{6}} = \frac{6\sqrt{6}-4}{29-6\sqrt{6}} \times \frac{29+6\sqrt{6}}{29+6\sqrt{6}}$$

$$= \frac{100+150\sqrt{6}}{625} = \frac{4+6\sqrt{6}}{25} \times \frac{4-6\sqrt{6}}{4-6\sqrt{6}}$$

$$= \frac{-200}{25\left(4-6\sqrt{6}\right)} = \frac{-8}{4-6\sqrt{6}} = \frac{4}{3\sqrt{6}-2}$$

16. If $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{99}+\sqrt{100}} = m$ and $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \ldots + \dfrac{1}{99\cdot 100} = n$, then the point (m, n) lies on the line

(1) $11(x-1) - 100(y-2) = 0$

(2) $11(x-2) - 100(y-1) = 0$

(3) $11(x-1) - 100y = 0$

(4) $11x - 100y = 0$

Ans. (4)

Sol. $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{99}+\sqrt{100}} = m$

$\dfrac{\sqrt{1}-\sqrt{2}}{-1} + \dfrac{\sqrt{2}-\sqrt{3}}{-1} \ldots \dfrac{\sqrt{99}-\sqrt{100}}{-1} = m$

$\sqrt{100} - 1 = m \Rightarrow m = 9$

$\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \ldots \dfrac{1}{99\cdot 100} = n$

$\dfrac{1}{1} - \dfrac{1}{2} + \dfrac{1}{2} - \dfrac{1}{3} \ldots \dfrac{1}{99} - \dfrac{1}{100} = n$

$1 - \dfrac{1}{100} = n$

$\dfrac{99}{100} = n$

$(m, n) = \left(9, \dfrac{99}{100}\right)$

$\Rightarrow 11(9) - 100\left(\dfrac{99}{100}\right)$

$= 99 - 99 = 0$

Ans. option (4) $11x - 100y = 0$

17. Let $f(x) = x^5 + 2x^3 + 3x + 1$, $x \in R$, and g(x) be a function such that $g(f(x)) = x$ for all $x \in R$. Then $\dfrac{g(7)}{g'(7)}$ is equal to :

(1) 7 (2) 42

(3) 1 (4) 14

Ans. (4)

Sol. $f(x) = x^5 + 2x^3 + 3x + 1$

$f'(x) = 5x^4 + 6x^2 + 3$

$f'(1) = 5 + 6 + 3 = 14$

$g(f(x)) = x$

$g'(f(x))f'(x) = 1$

for $f(x) = 7$

$\Rightarrow x^5 + 2x^3 + 3x + 1 = 7$

$\Rightarrow x = 1$

$g'(7)\, f'(1) = 1 \Rightarrow g'(7) = \dfrac{1}{f'(1)} = \dfrac{1}{14}$

$x = 1, f(x) = 7 \Rightarrow g(7) = 1$

$$\frac{g(7)}{g'(7)} = \frac{1}{1/14} = 14$$

18. If A(l, –1, 2), B(5, 7, –6), C(3, 4, –10) and D(–l, –4, –2) are the vertices of a quadrilateral ABCD, then its area is :

(1) $12\sqrt{29}$ (2) $24\sqrt{29}$

(3) $24\sqrt{7}$ (4) $48\sqrt{7}$

Ans. (1)

Sol. A(1, –1, 2)
B(5, 7, –6)
C(3, 4, –10)
D(–1, –4, –2)

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right| = \frac{1}{2}\left|\left(2\hat{i} + 5\hat{j} - 12\hat{k}\right) \times \left(6\hat{i} + 11\hat{j} - 4\hat{k}\right)\right|$$

$$= \frac{1}{2}\left|112\hat{i} - 64\hat{j} - 8\hat{k}\right|$$

$$= 4\left|14\hat{i} - 8\hat{j} - \hat{k}\right|$$

$$= 4\sqrt{196 + 64 + 1}$$

$$= 4\sqrt{261}$$

$$= 12\sqrt{29}$$

19. The value of $\int_{-\pi}^{\pi} \frac{2y(1+\sin y)}{1+\cos^2 y} dy$ is :

(1) π^2 (2) $\frac{\pi^2}{2}$

(3) $\frac{\pi}{2}$ (4) $2\pi^2$

Ans. (1)

Sol. $\int_{-\pi}^{\pi} \frac{2y(1+\sin y)}{1+\cos^2 y} dy$

$$= \int_{-\pi}^{\pi} \frac{2y}{1+\cos^2 y} dy + \int_{-\pi}^{\pi} \frac{2y\sin y}{1+\cos^2 y} dy$$

(Odd) (Even)

$$= 0 + 2.2\int_0^{\pi} y\left(\frac{\sin y}{1+\cos^2 y}\right) dy$$

$$I = 4\int_0^{\pi} \frac{y \sin y}{1+\cos^2 y} dy$$

$$I = 4\int_0^{\pi} \frac{(\pi - y)\sin y}{1+\cos^2 y} dy$$

$$2I = 4\int_0^{\pi} \frac{\pi \sin y}{1+\cos^2 y} dy$$

$$I = 2\pi\int_0^{\pi} \frac{\sin y}{1+\cos^2 y} dy$$

$$= 2\pi\left(-\tan^{-1}(\cos y)\right)_0^{\pi}$$

$$= -2\pi\left[\left(-\frac{\pi}{4}\right) - \left(\frac{\pi}{4}\right)\right]$$

$$= -2\pi\left[-\frac{2\pi}{4}\right] = \pi^2$$

20. If the line $\frac{2-x}{3} = \frac{3y-2}{4\lambda+1} = 4-z$ makes a right angle with the line $\frac{x+3}{3\mu} = \frac{1-2y}{6} = \frac{5-z}{7}$, then $4\lambda + 9\mu$ is equal to :

(1) 13 (2) 4

(3) 5 (4) 6

Ans. (4)

Sol. $\frac{2-x}{3} = \frac{3y-2}{4\lambda+1} = 4-z$ …(1)

$$\frac{x-2}{(-3)} = \frac{y-\frac{2}{3}}{\left(\frac{4\lambda+1}{3}\right)} = \frac{z-4}{(-1)}$$

$$\frac{x+3}{3\mu}=\frac{1-2y}{6}=\frac{5-z}{7} \quad \ldots(2)$$

$$\frac{x+3}{3\mu}=\frac{y-\frac{1}{2}}{(-3)}=\frac{z-5}{(-7)}$$

Right angle $\Rightarrow (-3)(3\mu)+\left(\frac{4\lambda+1}{3}\right)(-3)+(-1)(-7)=0$

$-9\mu-4\lambda-1+7=0$

$4\lambda+9\mu=6$

SECTION-B

21. From a lot of 10 items, which include 3 defective items, a sample of 5 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the variance of X is σ^2, then $96\sigma^2$ is equal to__________.

Ans. (56)

Sol. X = denotes number of defective

x	0	1	2	3
P(x)	$\frac{7}{15}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$
x_1^2	0	1	4	9
$P_i x_1^2$	0	$\frac{5}{12}$	$\frac{20}{12}$	$\frac{9}{12}$
$p_i x_i$	0	$\frac{5}{12}$	$\frac{10}{12}$	$\frac{3}{12}$

$$\mu=\Sigma p_i x_i=\frac{18}{12}$$

$$\Sigma p_i x_1^2=\frac{34}{12}$$

$$\sigma^2=\Sigma p_i x_1^2-(\mu)^2$$

$$=\frac{34}{12}-\left(\frac{18}{12}\right)^2=\frac{17}{6}-\frac{9}{4}$$

$$\frac{34-27}{12}=\frac{7}{12}$$

$$96\sigma^2=96\times\frac{7}{12}=56$$

22. If the constant term in the expansion of $(1+2x-3x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is p, then 108p is equal to

Ans. (54)

Sol. $\left(1+2x-3x^3\right)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

General term m $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

$$={}^9C_r\cdot\frac{3^{9-2r}}{2^{9-r}}(-1)^r\cdot x^{18-3r}$$

Put r = 6 to get coeff. of $x^0={}^9C_6\cdot\frac{1}{6^3}\cdot x^0=\frac{7}{18}x^0$

Put r = 7 to get coeff. of $x^{-3}={}^9C_r\cdot\frac{3^{-5}}{2^2}(-1)^7\cdot x^{-3}$

$$=-{}^9C_7\cdot\frac{1}{3^5\cdot 2^2}\cdot x^{-3}=\frac{-1}{27}x^{-3}$$

$$\left(1+2x-3x^3\right)\left(\frac{7}{18}x^0-\frac{1}{27}x^{-3}\right)$$

$$\frac{7}{18}+\frac{3}{27}=\frac{7}{18}+\frac{1}{9}=\frac{7+2}{18}=\frac{9}{18}=\frac{1}{2}$$

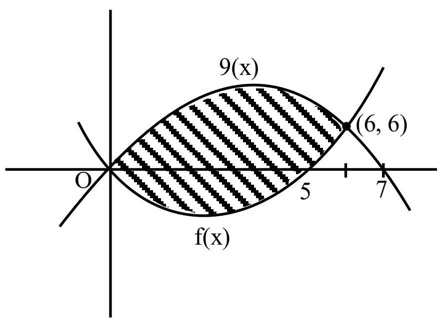
$$\therefore\ 108\cdot\frac{1}{2}=54$$

23. The area of the region enclosed by the parabolas $y=x^2-5x$ and $y=7x-x^2$ is __________.

Ans. (72)

NTA Ans. (198)

Sol. $y=x^2-5x$ and $y=7x-x^2$



$\int_0^6 (g(x)-f(x))dx$

$\int_0^6 \left((7x-x^2)-(x^2-5x)\right)dx$

$\int_0^6 \left(12x-2x^2\right)dx = \left[12\frac{x^2}{2}-\frac{2x^3}{3}\right]_0^6$

$\Rightarrow 6(6)^2 - \frac{2}{3}(6)^3$

$= 216 - 144 = 72 \text{ unit}^2$

24. The number of ways of getting a sum 16 on throwing a dice four times is ________.

Ans. (125)

Sol. $(x^1 + x^2 \ldots. + x^6)^4$

$x^4 \cdot \left(\frac{1-x^6}{1-x}\right)^4$

$x^4 \cdot (1-x^6)^4 \cdot (1-x)^{-4}$

$x^4[1-4x^6+6x^{12}\ldots.]\ [(1-x)^{-4}]$

$(x^4-4x^{10}+6x^{16}\ldots.)\ (1-x)^{-4}$

$(x^4-4x^{10}+6x^{16})\ (1+{}^{15}C_{12}x^{12}+{}^{9}C_6x^6\ \ldots.)$

$({}^{15}C_{12}-4\cdot{}^{9}C_6+6)x^{16}$

$({}^{15}C_3-4\cdot{}^{9}C_6+6)$

$= 35 \times 13 - 6 \times 8 \times 7 + 6$

$= 455 - 336 + 6$

$= 125$

25. If $S = \{a \in R : |2a-1| = 3[a]+2\{a\}\}$, where $[t]$ denotes the greatest integer less than or equal to t and $\{t\}$ represents the fractional part of t, then $72\sum_{a\in S} a$ is equal to ________.

Ans. (18)

Sol. $|2a-1| = 3[a]+2\{a\}$

$|2a-1| = [a]+2a$

Case-1 : $a > \frac{1}{2}$

$2a-1 = [a]+2a$

$[a] = -1 \quad \therefore a \in [-1, 0)$ Reject

Case-2 : $a < \frac{1}{2}$

$-2a+1 = [a]+2a$

$a = I + f$

$-2(I+f)+1 = I+2I+2f$

$I = 0,\ f = \frac{1}{4} \quad \therefore\ a = \frac{1}{4}$

Hence $a = \frac{1}{4}$

$72\sum_{a\in S} a = 72 \times \frac{1}{4} = 18$

26. Let f be a differentiable function in the interval $(0, \infty)$ such that $f(1) = 1$ and $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x} = 1$ for each $x > 0$. Then $2\ f(2) + 3\ f(3)$ is equal to ________.

Ans. (24)

Sol. $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x} = 1$

$\lim_{t\to x}\frac{2t.f(x)-x^2f'(x)}{1} = 1$

$2x.f(x) - x2f'(x) = 1$

$\frac{dy}{dx} - \frac{2}{x}\cdot y = \frac{-1}{x^2}$

$\text{I.f.} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$

$\therefore\ \frac{y}{x^2} = \int -\frac{1}{x^4}dx + C$

$\frac{y}{x^2} = \frac{1}{3x^3} + C$

Put $f(1) = 1$

$$C=\frac{2}{3}$$

$$y=\frac{1}{3x}+\frac{2x^2}{3}$$

$$y=\frac{2x^3+1}{3x}$$

$$f(2)=\frac{17}{6}$$

$$f(3)=\frac{55}{9}$$

$$2f(2)+3f(3)=\frac{17}{3}+\frac{55}{3}=\frac{72}{3}=24$$

27. Let $a_1, a_2, a_3, \ldots$ be in an arithmetic progression of positive terms.

Let $A_k = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2k-1}^2 - a_{2k}^2$.

If $A_3 = -153$, $A_5 = -435$ and $a_1^2 + a_2^2 + a_3^2 = 66$, then $a_{17} - A_7$ is equal to ___________.

Ans. (910)

Sol. $d \rightarrow$ common diff.

$A_k = -kd[2a + (2k-1)d]$

$A_3 = -153$

$\Rightarrow 153 = 13d[2a + 5d]$

$51 = d[2a + 5d]$ ...(1)

$A_5 = -435$

$435 = 5d[2a + 9d]$

$87 = d[2a + 9d]$

$(2) - (1)$

$36 = 4d^2$

$d = 3, a = 1$

$a_{17} - A_7 = 49 - [-7.3[2 + 39]] = 910$

28. Let $\vec{a}=\hat{i}-3\hat{j}+7\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}$ be a vector such that $(\vec{a}+2\vec{b})\times\vec{c}=3(\vec{c}\times\vec{a})$. If $\vec{a}\cdot\vec{c}=130$, then $\vec{b}\cdot\vec{c}$ is equal to ___________.

Ans. (30)

Sol. $(\vec{a}+2\vec{b})\times\vec{c}=3(\vec{c}\times\vec{a})$

$(2\vec{b}+4\vec{a})\times\vec{c}=0$

$\vec{c}=\lambda(4\vec{a}+2\vec{b})=\lambda(8\hat{i}-14\hat{j}+30\hat{k})$

$\vec{a}\cdot\vec{c}=130$

$8\lambda + 42\lambda + 210\lambda = 130$

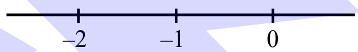
$\lambda=\frac{1}{2}$

$\vec{c}=4\hat{i}-7\hat{j}+15\hat{k}$

$\vec{b}\cdot\vec{c} = 8 + 7 + 15 = 30$

29. The number of distinct real roots of the equation $|x|\,|x+2| - 5|x+1| - 1 = 0$ is ___________.

Ans. (3)



Sol.

Case-1

$x \geq 0$

$x^2 + 2x - 5x - 5 - 1 = 0$

$x^2 - 3x - 6 = 0$

$$x=\frac{3\pm\sqrt{9+24}}{2}=\frac{3\pm\sqrt{33}}{2}$$

One positive root

Case-2

$-1 \leq x < 0$

$-x^2 - 2x - 5x - 5 - 1 = 0$

$x^2 + 7x + 6 = 0$

$(x + 6)(x + 1) = 0$

$x = -1$

one root in range

Case-3

$-2 \leq x < -1$

$x^2 - 2x + 5x + 5 - 1 = 0$

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

No root in range

Case-4

$x < -2$

$x^2 + 7x + 4 = 0$

$$x = \frac{-7 \pm \sqrt{49-16}}{2} = \frac{7 \pm \sqrt{33}}{2}$$

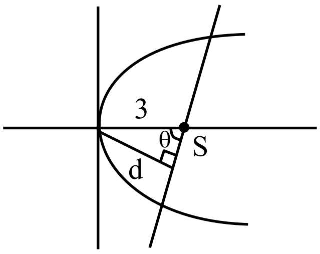
one root in range

Total number of distinct roots are 3

30. Suppose AB is a focal chord of the parabola $y^2 = 12x$ of length l and slope $m < \sqrt{3}$. If the distance of the chord AB from the origin is d, then ld^2 is equal to __________.

Ans. (108)

Sol.



$\ell = 4a\ \text{cosec}^2\theta$

$$\ell = 12 \times \frac{9}{d^2}$$

$\ell d^2 = 108$