FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Thursday 04th April, 2024) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. If the function $f(x)=\begin{cases}\dfrac{72^x-9^x-8^x+1}{\sqrt{2}-\sqrt{1+\cos x}} & , x\neq 0\\ a\log_e 2\log_e 3 & , x=0\end{cases}$

is continuous at x = 0, then the value of a^2 is equal to

(1) 968 (2) 1152

(3) 746 (4) 1250

Ans. (2)

Sol. $\lim\limits_{x\to 0} f(x)=a\ell n2\ell n3$

$$\lim_{n\to 0}\frac{72^x-9^x-8^x+1}{\sqrt{2}-\sqrt{1+\cos x}}=\lim_{x\to 0}\frac{(8^x-1)(9^x-1)}{\sqrt{2}-\sqrt{1+\cos x}}$$

$$\lim_{n\to 0}\left(\frac{8^x-1}{x}\right)\left(\frac{9^x-1}{x}\right)\left(\frac{x^2}{1-\cos x}\right)\left(\sqrt{2}+\sqrt{1+\cos x}\right)$$

$\therefore\ \ell n8\times\ell n9\times 2\times 2\sqrt{2}=24\sqrt{2}\ell n2\ell n3$

$\therefore$ a = $24\sqrt{2}$, $a^2 = 576 \times 2 = 1152$

2. If λ > 0, let θ be the angle between the vectors $\vec{a}=\hat{i}+\lambda\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$. If the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are mutually perpendicular, then the value of $(14\cos\theta)^2$ is equal to

(1) 25 (2) 20

(3) 50 (4) 40

Ans. (1)

Sol. $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$, $\lambda>0$

$|\vec{a}|^2-|\vec{b}|^2=0 \rightarrow 1+\lambda^2+9=9+1+4$

$\therefore \lambda=2,\ \cos\theta=\dfrac{\vec{a}-\vec{b}}{|\vec{a}|.|\vec{b}|}=\dfrac{3-\lambda-6}{\sqrt{14}.\sqrt{14}}$

$14\cos\theta = 3 - 8 = -5$

$\therefore (14\cos\theta)^2 = 25$

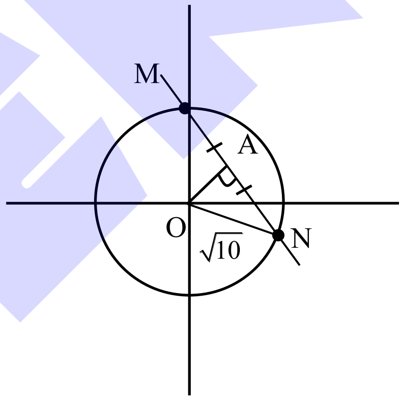
3. Let C be a circle with radius $\sqrt{10}$ units and centre at the origin. Let the line x + y = 2 intersects the circle C at the points P and Q. Let MN be a chord of C of length 2 unit and slope –1. Then, a distance (in units) between the chord PQ and the chord MN is

(1) $2-\sqrt{3}$ (2) $3-\sqrt{2}$

(3) $\sqrt{2}-1$ (4) $\sqrt{2}+1$

Ans. (2)

Allen Ans. ( )



$C : x^2 + y^2 = 10$

$AN = \dfrac{MN}{2} = 1$

$\therefore$ In $\Delta OAN \rightarrow (ON)^2 = (OA)^2 + (AN)^2$

$10 = (OA)^2 + 1 \rightarrow OA = 3$

Perpendicular distance of center from

$PQ=\dfrac{|0+0-2|}{\sqrt{2}}=\sqrt{2}$

Perpendicular distance between MN and

$PQ=OA+\sqrt{2}$ or $|OA-\sqrt{2}|$

$=3+\sqrt{2}$ or $3-\sqrt{2}$

4. Let a relation R on $\mathbb{N}\times\mathbb{N}$ be defined as :

(x_1,y_1) R(x_2,y_2) if and only if $x_1 \le x_2$ or $y_1 \le y_2$

Consider the two statements :

(I) R is reflexive but not symmetric.

(II) R is transitive

Then which one of the following is true ?

(1) Only (II) is correct.

(2) Only (I) is correct.

(3) Both (I) and (II) are correct.

(4) Neither (I) nor (II) is correct.

Ans. (2)

Sol. All $((x_1y_1), (x_1,y_1))$ are in R where

$x_1, y_1 \in N \therefore$ R is reflexive

$((1,1), (2,3)) \in R$ but $((2,3), (1,1)) \notin R$

$\therefore$ R is not symmetric

$((2,4), (3,3)) \in R$ and $((3,3), (1,3)) \in R$ but $((2,4), (1,3)) \notin R$

$\therefore$ R is not transitive

5. Let three real numbers a,b,c be in arithmetic progression and a + 1, b, c + 3 be in geometric progression. If a > 10 and the arithmetic mean of a,b and c is 8, then the cube of the geometric mean of a,b and c is

(1) 120 (2) 312

(3) 316 (4) 128

Ans. (1)

Sol. $2b = a + c,\ b^2 = (a + 1)(c + 3),$

$$\frac{a+b+c}{3}=8\rightarrow b=8, a+c=16$$

$64 = (a + 1)(19 - a) = 19 + 18a - a^2$

$a^2 - 18a - 45 = 0 \rightarrow (a - 15)(a + 3) = 0,\ (a > 10)$

$a = 15,\ c = 1,\ b = 8$

$((abc)^{1/3})^3 = abc = 120$

6. Let $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ and $B = I + adj(A) + (adj\ A)^2+\ldots+(adj\ A)^{10}$. Then, the sum of all the elements of the matrix B is :

(1) –110 (2) 22

(3) –88 (4) –124

Ans. (3)

Sol. $Adj(A)=\begin{bmatrix}1&-2\\0&1\end{bmatrix}$

$$(AdjA)^2=\begin{bmatrix}1&-4\\0&1\end{bmatrix}$$

|

|

$$(AdjA)^{10}=\begin{bmatrix}1&-20\\0&1\end{bmatrix}$$

$$B=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}1&-2\\0&1\end{bmatrix}+\begin{bmatrix}1&-4\\0&1\end{bmatrix}+\ldots+\begin{bmatrix}1&-20\\0&1\end{bmatrix}$$

$B=\begin{bmatrix}11&-110\\0&11\end{bmatrix} \Rightarrow$ sum of elements of B

$= -88$

7. The value of $\frac{1\times2^2+2\times3^2+\ldots+100\times(101)^2}{1^2\times2+2^2\times3+\ldots+100^2\times101}$ is

(1) $\frac{306}{305}$ (2) $\frac{305}{301}$

(3) $\frac{32}{31}$ (4) $\frac{31}{30}$

Ans. (2)

Sol. $$\frac{1\times2^2+2\times3^2+\ldots+100\times(101)^2}{1^2\times2+2^2\times3+\ldots+100^2\times101}=\frac{\sum_{r=1}^{100}r(r+1)^2}{\sum_{r=1}^{100}r^2(r+1)}$$

$$=\frac{\sum_{r=1}^{100}(r^3+2r^2+r)}{\sum_{r=1}^{100}(r^3+r^2)}=\frac{\left(\frac{n(n+1)}{2}\right)^2+\frac{2.n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}}{\left(\frac{n(n+1)}{2}\right)^2+\frac{n(n+1)(2n+1)}{6}}$$

$$=\frac{\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}.(2n+1)+1\right]}{\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{(2n+1)}{3}\right]};\text{Put } n=100$$

$$=\frac{\frac{100(101)}{2}+\frac{2}{3}(201)+1}{\frac{100\times101}{2}+\frac{201}{3}}=\frac{5185}{5117}=\frac{305}{301}$$

8. Let $f(x)=\int_0^x\left(t+\sin\left(1-e^t\right)\right)dt, x\in\mathbb{R}$.

Then $\lim_{x\to 0}\frac{f(x)}{x^3}$ is equal to

(1) $\frac{1}{6}$ (2) $-\frac{1}{6}$

(3) $-\frac{2}{3}$ (4) $\frac{2}{3}$

Ans. (2)

Sol. $\lim_{x\to 0}\frac{f(x)}{x^3}$

Using L Hopital Rule.

$\lim_{x\to 0}\frac{f'(x)}{3x^2}=\lim_{x\to 0}\frac{x+\sin\left(1-e^x\right)}{3x^2}$ (Again L Hopital)

Using L.H. Rule

$$=\lim_{x\to 0}\frac{-\left[\sin\left(1-e^x\right)\left(-e^x\right).e^x+\cos\left(1-e^x\right).e^x\right]}{6}$$

$$=-\frac{1}{6}$$

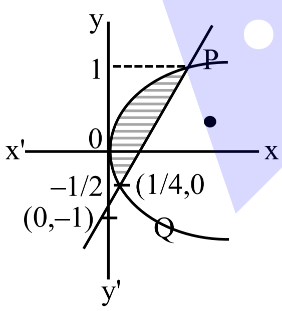
9. The area (in sq. units) of the region described by $\{(x,y) : y^2\le 2x, \text{ and } y\ge 4x-1\}$ is

(1) $\frac{11}{32}$ (2) $\frac{8}{9}$

(3) $\frac{11}{12}$ (4) $\frac{9}{32}$

Ans. (4)

Sol.



Shaded area $=\int_{-\frac{1}{2}}^{1}\left(x_{Right}-x_{Left}\right)dy$

$y^2=2x$
$y=4x-1$ Solve
$y=1, y=-\frac{1}{2}$

Shaded area $=\int_{-\frac{1}{2}}^{1}\left(\frac{y+1}{4}-\frac{y^2}{2}\right)dy$

$$=\left(\frac{1}{4}\left(\frac{y^2}{2}+y\right)-\frac{y^3}{6}\right)_{-\frac{1}{2}}^{1}=\frac{9}{32}$$

10. The area (in sq. units) of the region $S=\{z\in\mathbb{C};|z-1|\le 2;(z+\bar{z})+i(z-\bar{z})\le 2, \text{Im}(z)\ge 0\}$ is

(1) $\frac{7\pi}{3}$ (2) $\frac{3\pi}{2}$

(3) $\frac{17\pi}{8}$ (4) $\frac{7\pi}{4}$

Ans. (2)

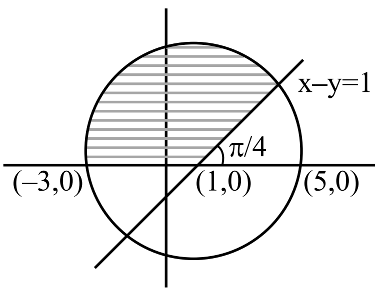
Sol. Put $z = x + iy$

$|z-1|\le 2\Rightarrow(x-1)^2+y^2\le 4$ ...(1)

$(z+\bar{z})+i(z-\bar{z})\le 2\Rightarrow 2x+i(2iy)\le 2$

$\Rightarrow x-y\le 1$ ...(2)

$\text{Im}(z)\ge 0\Rightarrow y\ge 0$ ...(3)



Required area

= Area of semi-circle – area of sector A

$\frac{1}{2}\pi(2)^2-\frac{\pi}{2}$

$=\frac{3\pi}{2}$

11. If the value of the integral $\int_{-1}^{1}\frac{\cos\alpha x}{1+3^x}dx$ is $\frac{2}{\pi}$.

Then, a value of α is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

Ans. (2)

Sol. Let $I=\int_{-1}^{+1}\frac{\cos\alpha x}{1+3^x}dx$ ...(I)

$I=\int_{-1}^{+1}\frac{\cos\alpha x}{1+3^{-x}}dx$

$\left(\text{using}\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$ ...(II)

Add (1) and (II)

$2I=\int_{-1}^{+1}\cos(\alpha x)dx=2\int_0^1\cos(\alpha x)dx$

$I=\frac{\sin\alpha}{\alpha}=\frac{2}{\pi}(\text{given})$

$\therefore\ \alpha=\frac{\pi}{2}$

12. Let $f(x)=3\sqrt{x-2}+\sqrt{4-x}$ be a real valued function. If α and β are respectively the minimum and the maximum values of f, then $\alpha^2+2\beta^2$ is equal to

(1) 44 (2) 42

(3) 24 (4) 38

Ans. (2)

Sol. $f(x)=3\sqrt{x-2}+\sqrt{4-x}$

$x-2\geq 0\ \&\ 4-x\geq 0$

$\therefore x\in[2,4]$

Let $x=2\sin^2\theta+4\cos^2\theta$

$\therefore f(x)=3\sqrt{2}|\cos\theta|+\sqrt{2}|\sin\theta|$

$\therefore \sqrt{2}\leq 3\sqrt{2}|\cos\theta|+\sqrt{2}|\sin\theta|\leq\sqrt{9\times 2+2}$

$\sqrt{2}\leq 3\sqrt{2}|\cos\theta|+\sqrt{2}|\sin\theta|\leq\sqrt{20}$

$\therefore \alpha=\sqrt{2}\quad \beta=\sqrt{20}$

$\alpha^2+2\beta^2=2+40=42$

13. If the coefficients of x^4, x^5 and x^6 in the expansion of $(1+x)^n$ are in the arithmetic progression, then the maximum value of n is :

(1) 14 (2) 21

(3) 28 (4) 7

Ans. (1)

Sol. Coeff. of $x^4={}^nC_4$

Coeff. of $x^5={}^nC_5$

Coeff. of $x^6={}^nC_6$

${}^nC_4, {}^nC_5, {}^nC_6$ .... AP

$2.{}^nC_5={}^nC_4+{}^nC_6$

$2=\frac{{}^nC_4}{{}^nC_5}+\frac{{}^nC_6}{{}^nC_5}\quad\left\{\frac{{}^nC_r}{{}^nC_{r-1}}=\frac{n-r+1}{r}\right\}$

$2=\frac{5}{n-4}+\frac{n-5}{6}$

$12(n-4)=30+n^2-9n+20$

$n^2-21n+98=0$

$(n-14)(n-7)=0$

$n_{max}=14\quad n_{min}=7$

14. Consider a hyperbola H having centre at the origin and foci and the x-axis. Let C_1 be the circle touching the hyperbola H and having the centre at the origin. Let C_2 be the circle touching the hyperbola H at its vertex and having the centre at one of its foci. If areas (in sq. units) of C_1 and C_2 are 36π and 4π, respectively, then the length (in units) of latus rectum of H is

(1) $\frac{28}{3}$ (2) $\frac{14}{3}$

(3) $\frac{10}{3}$ (4) $\frac{11}{3}$

Ans. (1)

Sol. Let H : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\qquad (b^2 = a^2(e^2 - 1))$

$\therefore$ eq^n of $C_1 = x^2 + y^2 = a^2$

Ar. = 36π

$\pi a^2 = 36\pi$

$a = 6$

Now radius of C_2 can be $a(e - 1)$ or $a(e + 1)$

for $r = a(e - 1)$	for $r = a(e + 1)$
Ar. = 4π	$\pi r^2 = 4\pi$
$\pi a^2(e-1)^2 = 4\pi$	$a^2(e+1)^2 = 4$
$36\pi(e-1)^2 = 4\pi$	$36(e+1)^2 = 4$
$e - 1 = \frac{1}{3}$	$e + 1 = \frac{1}{3}$
$e = \frac{4}{3}$	$-\frac{2}{3}$
	Not possible

$\therefore b^2 = 36\left(\frac{16}{9}-1\right) = 28$

$\therefore LR = \frac{2b^2}{a} = \frac{2\times 28}{6} = \frac{28}{3}$

15. If the mean of the following probability distribution of a random variable X;

X	0	2	4	6	8
P(X)	a	2a	a + b	2b	3b

is $\frac{46}{9}$, then the variance of the distribution is

(1) $\frac{581}{81}$ (2) $\frac{566}{81}$

(3) $\frac{173}{27}$ (4) $\frac{151}{27}$

Ans. (2)

Sol. $\sum P_i = 1$

$a + 2a + a + b + 2b + 3b = 1$

$4a + 6b = 1$ .... (I)

$E(x) = \text{mean} = \frac{46}{9}$

$\sum P_i X_i = \frac{46}{9} \Rightarrow 4a + 4a + 4b + 12b + 24b = \frac{46}{9}$

$8a + 40b = \frac{46}{9}$

$4a + 20b = \frac{23}{9}$ .... (II)

Subtract (I) from (II) we get

$b = \frac{1}{9}$ & $a = \frac{1}{12}$

Variance $= E(x_i^2) - E(x_i)^2$

$E(x_i^2) = 0^2 \times 9^2 + 2^2 \times 2a + 4^2(a + b) + 6^2(2b) + 8^2(3b)$

$= 24a + 280b$

Put $a = \frac{1}{12}$ $\quad b = \frac{1}{9}$

$E(x_i^2) = 2 + \frac{280}{9} = \frac{298}{9}$

$\therefore \sigma^2 = E(x_i^2) - E(x_i)^2$

$= \frac{298}{9} - \left(\frac{46}{9}\right)^2$

$\sigma^2 = \frac{298}{9} - \frac{2116}{81}$

$= \frac{566}{81}$

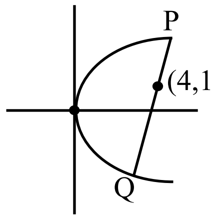
16. Let PQ be a chord of the parabola $y^2 = 12x$ and the midpoint of PQ be at (4,1). Then, which of the following point lies on the line passing through the points P and Q ?

(1) (3,–3) (2) $\left(\frac{3}{2},-16\right)$

(3) (2,–9) (4) $\left(\frac{1}{2},-20\right)$

Ans. (4)

Sol.



$T = S_1$

$y - 6(x + 4)$

$= 1 - 48$

$6x - y = 23$

Option 4 $\left(\frac{1}{2},-20\right)$ will satisfy

17. Given the inverse trigonometric function assumes principal values only. Let x, y be any two real numbers in [–1,1] such that

$\cos^{-1}x - \sin^{-1}y = \alpha, \frac{-\pi}{2} \le \alpha \le \pi$.

Then, the minimum value of $x^2 + y^2 + 2xy\sin\alpha$ is

(1) –1 (2) 0

(3) $\frac{-1}{2}$ (4) $\frac{1}{2}$

Ans. (2)

Sol. $\cos^{-1}x - \left(\frac{\pi}{2} - \cos^{-1}y\right) = \alpha$

$\cos^{-1}x + \cos^{-1}y = \frac{\pi}{2} + \alpha$

$\alpha \in \left[-\frac{\pi}{2}, \pi\right], \frac{\pi}{2} + \alpha \in \left[0, \frac{3\pi}{2}\right]$

$\cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \frac{\pi}{2} + \alpha$

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\sin\alpha$

$(xy + \sin\alpha) = (1 - x^2)(1 - y^2)$

$x^2y^2 + 2xy\sin a + \sin^2 a = 1 - x^2 - y^2 + x^2y^2$

$x^2 + y^2 + 2xy\sin\alpha = 1 - \sin^2\alpha$

$x^2 + y^2 + 2xy\sin\alpha = \cos^2\alpha$

Min. value of $\cos^2\alpha = 0$

At $\boxed{\alpha = \frac{\pi}{2}}$

Option (2) is correct

18. Let y = y(x) be the solution of the differential equation $(x^2 + 4)^2dy + (2x^3y + 8xy - 2)dx = 0$. If y(0) = 0, then y(2) is equal to

(1) $\frac{\pi}{8}$ (2) $\frac{\pi}{16}$

(3) 2π (4) $\frac{\pi}{32}$

Ans. (4)

Sol. $\frac{dy}{dx} + y\left(\frac{2x^3 + 8x}{(x^2+4)^2}\right) = \frac{2}{(x^2+4)^2}$

$\frac{dy}{dx} + y\left(\frac{2x}{x^2+4}\right) = \frac{2}{(x^2+4)^2}$

$IF = e^{\int \frac{2x}{x^2+4}dx}$

$IF = x^2 + 4$

$y \times (x^2 + 4) = \int \frac{2}{(x^2+4)^2} \times (x^2+4)$

$y(x^2 + 4) = 2\int \frac{dx}{x^2 + 2^2}$

$y(x^2+4) = \frac{2}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$0 = 0 + c = c = 0$

$y(x^2 + 4) = \tan^{-1}\left(\frac{x}{2}\right)$

y at x = 2

$y(4 + 4) = \tan^{-1}(1)$

$\boxed{y(2) = \frac{\pi}{32}}$

Option (4) is correct

19. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = x\hat{i} + 2\hat{j} + 3\hat{k}, x \in \mathbb{R}$. If $\vec{d}$ is the unit vector in the direction of $\vec{b} + \vec{c}$ such that $\vec{a}.\vec{d} = 1$, then $(\vec{a} \times \vec{b}).\vec{c}$ is equal to

(1) 9 (2) 6

(3) 3 (4) 11

Ans. (4)

Sol. $\vec{d} = \lambda(\vec{b} + \vec{c})$

$\vec{a}.\vec{d} = \lambda(\vec{b}.\vec{a} + \vec{c}.\vec{a})$

$1 = \lambda(1 + x + 5)$

$1 = \lambda(x + 6)$ ....(1)

$|\vec{d}| = 1$ $\boxed{\frac{1}{\lambda} = x + 6}$

$|\lambda(\vec{b} + \vec{c})| = 1$

$|\lambda((x+2)\hat{i} + 6\hat{j} - 2\hat{k})| = 1$

$\lambda^2((x + 2)^2 + 6^2 + 2^2) = 1$

$x^2 + 4x + 4 + 36 + 4 = (x + 6)^2$

$x^2 + 4x + 44 = x^2 + 12x + 36$

$8x = 8, x = 1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -5 \\ x & 2 & 3 \end{vmatrix} = (\vec{a} \times \vec{b}).\vec{c}$$

$$\begin{vmatrix} 0 & 0 & 1 \\ -2 & 9 & -4 \\ x-2 & -1 & 3 \end{vmatrix} = 2 - 9(x - 2)$$

$= 20 - 9x$

at $x = 1$

$20 - 9 = 11$

Option 4 is correct

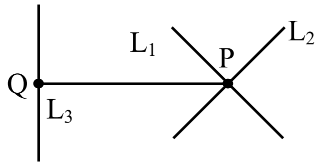
20. Let P the point of intersection of the lines $\frac{x-2}{1} = \frac{y-4}{5} = \frac{z-2}{1}$ and $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-3}{2}$.

Then, the shortest distance of P from the line $4x = 2y = z$ is

(1) $\frac{5\sqrt{14}}{7}$ (2) $\frac{\sqrt{14}}{7}$

(3) $\frac{3\sqrt{14}}{7}$ (4) $\frac{6\sqrt{14}}{7}$

Ans. (3)



$L_1 \equiv \frac{x-2}{1} = \frac{y-4}{5} = \frac{z-2}{1} = \lambda$

$P(\lambda+2, 5\lambda+4, \lambda+2)$

$L_2 \equiv \frac{x-3}{2} = \frac{y-2}{3} = \frac{z-3}{2}$

$P(2\mu+3, 3\mu+2, 2\mu+3)$

$\lambda + 2 = 2\mu + 3$ $3\mu + 2 = 5\lambda + 4$

$\lambda = 2\mu + 1$ $3\mu = 5\lambda + 2$

$3\mu = 5(2\mu + 1) + 2$

$3\mu = 10\mu + 7$

$\mu = -1$ $\lambda = -1$

Both satisfies (P)

P(1,–1,1)

$L_3 \equiv \frac{x}{1/4} = \frac{y}{1/2} = \frac{z}{1}$

$L_3 = \frac{x}{1} = \frac{y}{2} = \frac{z}{4} = k$

Coordinates of Q(k,2k,4k)

DR's of PQ = <k–1, 2k + 1, 4k – 1>

PQ $\perp$ to L_3

$(k - 1) + 2(2k + 1) + 4(4k - 1) = 0$

$k - 1 + 4k + 2 + 16k - 4 = 0$

$k = \frac{1}{7}$

$Q\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

$$PQ = \sqrt{\left(1-\frac{1}{7}\right)^2 + \left(-1-\frac{2}{7}\right)^2 + \left(1-\frac{4}{7}\right)^2}$$

$$= \sqrt{\frac{36}{49} + \frac{81}{49} + \frac{9}{49}} = \frac{\sqrt{126}}{7}$$

$$PQ = \frac{3\sqrt{14}}{7}$$

Option-3 will satisfy

SECTION-B

21. Let S = {$\sin^2 2\theta$: $(\sin^4\theta + \cos^4\theta)x^2 + (\sin 2\theta)x + (\sin^6\theta + \cos^6\theta) = 0$ has real roots}. If α and β be the smallest and largest elements of the set S, respectively, then $3((\alpha-2)^2 + (\beta-1)^2)$ equals.....

Ans. (4)

Sol. $D = (\sin 2\theta)^2 - 4\left(1 - \frac{\sin^2 2\theta}{2}\right)\left(1 - \frac{3}{4}\sin^2 2\theta\right)$

$= (\sin 2\theta)^2 - 4\left(1 - \frac{5}{4}\sin^2 2\theta + \frac{3}{8}\sin^4 2\theta\right)$

$D = -\frac{3}{2}\sin^4 2\theta + 6\sin^2 2\theta - 4 > 0$

$3\sin^4 2\theta - 12\sin^2 2\theta + 8 < 0$

$\sin^2 2\theta = \frac{12 \pm \sqrt{12^2 - 12.8}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = \frac{6 \pm 2\sqrt{3}}{3}$

$\sin^2 2\theta = 2 \pm \frac{2}{\sqrt{3}}$, but $\sin^2 2\theta \in [0, 1]$

$\therefore\ \alpha = 2 - \frac{2}{\sqrt{3}},\ \beta = 1 \rightarrow (\alpha-2)^2 = \frac{4}{3},\ (\beta-1)^2 = 0$

$\boxed{3(\alpha-2)^2 + (\beta-1)^2 = 4}$

22. If $\int \text{cosec}^5 x\,dx = \alpha \cot x\,\text{cosec}x\left(\text{cosec}^2 x + \frac{3}{2}\right) + \beta \log_e\left|\tan\frac{x}{2}\right| + C$

where $\alpha, \beta \in \mathbb{R}$ and C is constant of integration, then the value of $8(\alpha + \beta)$ equals .....

Ans. (1)

Sol. $\int \text{cosec}^3 x \cdot \text{cosec}^2 x\,dx = I$

By applying integration by parts

$I = -\cot x\,\text{cosec}^3 x + \int \cot x\left(-3\text{cosec}^2 x \cot x\,\text{cosec}x\right)dx$

$I = -\cot x\,\text{cosec}^3 x - 3\int \text{cosec}^3 x\left(\text{cosec}^2 x - 1\right)dx$

$I = -\cot x\,\text{cosec}^3 x - 3I + 3\int \text{cosec}^3 x\,dx$

let

$I_1 = \int \text{cosec}^3 x\,dx = -\text{cosec}x \cot x - \int \cot^2 x\,\text{cosec}x\,dx$

$I_1 = -\text{cosec}x\cot x - \int\left(\text{cosec}^2 x - 1\right)\text{cosec}x\,dx$

$2I_1 = -\text{cosec}x\cot x + \ell n\left|\tan\frac{x}{2}\right|$

$I_1 = -\frac{1}{2}\text{cosec}x\cot x + \frac{1}{2}\ell n\left|\tan\frac{x}{2}\right|$

$4I = -\cot x\,\text{cosec}^3 x - \frac{3}{2}\text{cosec}x\cot x + \frac{3}{2}\ell n\left|\tan\frac{x}{2}\right| + 4c$

$I = -\frac{1}{4}\text{cosec}x\cot x\left(\text{cosec}^2 x + \frac{3}{2}\right) + \frac{3}{8}\ell n\left|\tan\frac{x}{2}\right| + c$

$\therefore\ \alpha = \frac{-1}{4},\ \beta = \frac{3}{8} \rightarrow \boxed{8(\alpha+\beta) = 1}$

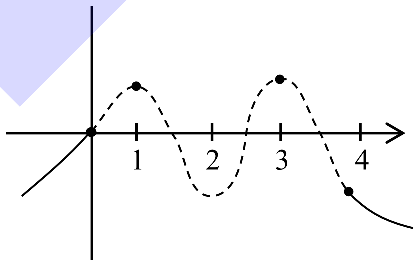
23. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a thrice differentiable function such that f(0) = 0, f(1) = 1, f(2) = –1, f(3) = 2 and f(4) = –2. Then, the minimum number of zeros of $(3f'f'' + ff''')(x)$ is .....

Ans. (5)

Sol. $(3f'f'' + ff''')(x) = \left(\left(ff'' + (f')^2\right)(x)\right)'$

$\left(ff'' + (f')^2\right)(x) = ((ff')(x))'$

$\therefore\ (3f'f'' + f''')(x) = (f(x)\cdot f'(x))''$



min. roots of f(x) $\rightarrow$ 4

$\therefore$ min. roots of f '(x) $\rightarrow$ 3

$\therefore$ min. roots of $(f(x)\cdot f'(x)) \rightarrow 7$

$\boxed{\therefore\ \text{min. roots of } (f(x)\cdot f'(x))'' \rightarrow 5}$

24. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$f(x) = \frac{2x}{\sqrt{1+9x^2}}$. If the composition of

$f,\underbrace{(f \text{ o } f \text{ o } f \text{ o}...\text{o } f)}_{10\,\text{times}}(x) = \frac{2^{10}x}{\sqrt{1+9\alpha x^2}}$, then the value of $\sqrt{3\alpha+1}$ is equal to .....

Ans. (1024)

Sol. $f(f(x)) = \frac{2f(x)}{\sqrt{1+9f^2(x)}} = \frac{4x}{\sqrt{1+9x^2+9.2^2x^2}}$

$f(f(f(x))) = \frac{2^3x/\sqrt{1+9x^2}}{\sqrt{1+9(1+2^2)\frac{2^2x^2}{1+9x^2}}} = \frac{2^3x}{\sqrt{1+9x^2(1+2^2+2^4)}}$

$\therefore$ By observation

$\alpha = 1+2^2+2^4+....+2^{18} = 1\left(\frac{(2^2)^{10}-1}{2^2-1}\right) = \frac{2^{20}-1}{3}$

$3\alpha+1 = 2^{20} \rightarrow \sqrt{3\alpha+1} = 2^{10} = \boxed{1024}$

25. Let A be a 2 × 2 symmetric matrix such that $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\7\end{bmatrix}$ and the determinant of A be 1.

If $A^{-1} = \alpha A + \beta I$, where I is an identity matrix of order 2 × 2, then $\alpha + \beta$ equals …..

Ans. (5)

Sol. Let $A = \begin{bmatrix}a & b\\b & d\end{bmatrix}$

$\begin{bmatrix}a & b\\b & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\7\end{bmatrix}, ad - b^2 = 1$

$a + b = 3,\ b + d = 7,\ (3 - b)(7 - b) - b^2 = 1$

$21 - 10b = 1 \rightarrow b = 2,\ a = 1,\ d = 5$

$A = \begin{bmatrix}1 & 2\\2 & 5\end{bmatrix}, A^{-1} = \begin{bmatrix}5 & -2\\-2 & 1\end{bmatrix}$

$A^{-1} = \alpha A + \beta I$

$\begin{bmatrix}5 & -2\\-2 & 1\end{bmatrix} = \begin{bmatrix}\alpha+\beta & 2\alpha\\2\alpha & 5\alpha+\beta\end{bmatrix}$

$\alpha = -1, \beta = 6 \rightarrow \boxed{\alpha+\beta = 5}$

26. There are 4 men and 5 women in Group A, and 5 men and 4 women in Group B. If 4 persons are selected from each group, then the number of ways of selecting 4 men and 4 women is …..

Ans. (5626)

Sol.

From Group A	From Group B	Ways of selection
4M	4W	${}^4C_4\,{}^4C_4 = 1$
3M 1W	1M 3W	${}^4C_3\,{}^5C_1\,{}^5C_1\,{}^4C_3 = 400$
2M 2W	2M 2W	${}^4C_2\,{}^5C_2\,{}^5C_2\,{}^4C_2 = 3600$
1M 3W	3M 1W	${}^4C_1\,{}^5C_3\,{}^5C_3\,{}^4C_1 = 1600$
4W	4M	${}^5C_4\,{}^5C_4 = 25$
Total		5626

Ans. 5626

27. In a tournament, a team plays 10 matches with probabilities of winning and losing each match as $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Let x be the number of matches that the team wins, and y be the number of matches that team loses. If the probability $P(|x - y| \le 2)$ is p, then 3^9p equals……

Ans. (8288)

Sol. $P(W) = \frac{1}{3}$ $\qquad P(L) = \frac{2}{3}$

x = number of matches that team wins

y = number of matches that team loses

$|x-y| \le 2$ and $x + y = 10$

$|x-y| = 0,1,2 \quad x, y \in N$

Case-I : $|x-y| = 0 \Rightarrow x = y$

$\because\ x+y = 10 \Rightarrow x = 5 = y$

$P(|x-y|=0) = {}^{10}C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5$

Case-II : $|x-y| = 1 \Rightarrow x - y = \pm 1$

$x = y+1$	$x = y-1$
$\because\ x+y=10$	$\because\ x+y=10$
$2y = 9$	$2y = 11$
Not possible	Not possible

Case-III : $|x-y|=2 \Rightarrow x-y=\pm 2$

$x-y=2$ OR $x-y=-2$

$\because x+y=10$ $\quad\because x+y=10$

$x=6, y=4$ $\quad x=4, y=6$

$$P(|x-y|=2)={}^{10}C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^4+{}^{10}C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6$$

$$p={}^{10}C_5\frac{2^5}{3^{10}}+{}^{10}C_6\frac{2^4}{3^{10}}+{}^{10}C_4\frac{2^6}{3^{10}}$$

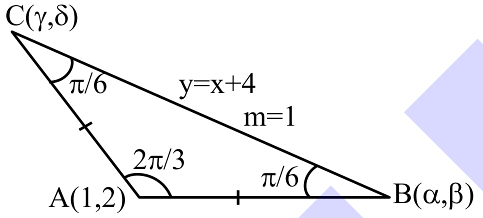
$$3^9 p=\frac{1}{3}\left({}^{10}C_5 2^5+{}^{10}C_6 2^4+{}^{10}C_4 2^6\right)$$

= 8288

28. Consider a triangle ABC having the vertices A(1,2), B(α,β) and C(γ,δ) and angles $\angle ABC=\frac{\pi}{6}$ and $\angle BAC=\frac{2\pi}{3}$. If the points B and C lie on the line y = x + 4, then $\alpha^2+\gamma^2$ is equal to …..

Ans. (14)

Sol.



Equation of line passes through point A(1, 2) which makes angle $\frac{\pi}{6}$ from y = x + 4 is

$$y-2=\frac{1\pm\tan\frac{\pi}{6}}{1\mp\tan\frac{\pi}{6}}(x-1)$$

$$y-2=\frac{\sqrt{3}\pm 1}{\sqrt{3}\mp 1}(x-1)$$

⊕

$y-2=(2+\sqrt{3})(x-1)$

solve with y = x + 4

$x+2=(2+\sqrt{3})x-2-\sqrt{3}$

$x=\frac{4+\sqrt{3}}{1+\sqrt{3}}$

⊖

$y-2=(2-\sqrt{3})(x-1)$

solve with y = x + 4

$x+2=(2-\sqrt{3})x-2+\sqrt{3}$

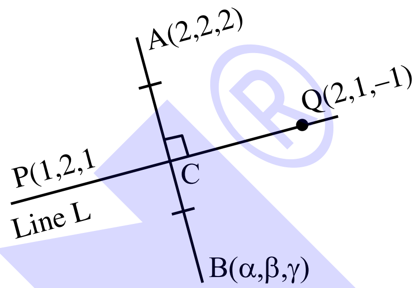
$x=\frac{4-\sqrt{3}}{1-\sqrt{3}}$

$$\alpha^2+\gamma^2=\left(\frac{4+\sqrt{3}}{1+\sqrt{3}}\right)^2+\left(\frac{4-\sqrt{3}}{1-\sqrt{3}}\right)^2$$

$$\alpha^2+\gamma^2=14$$

29. Consider a line L passing through the points P(1,2,1) and Q(2,1,–1). If the mirror image of the point A(2,2,2) in the line L is (α,β,γ), then α + β + 6γ is equal to …..

Ans. (6)



DR's of Line L ≡ –1 : 1 : 2

DR's of AB ≡ α – 2 : β – 2 : γ – 2

$AB \perp_{ar} L \Rightarrow 2-\alpha+\beta-2+2\gamma-4=0$

$2\gamma+\beta-\alpha=4$ ...(1)

Let C is mid-point of AB

$$C\left(\frac{\alpha+2}{2},\frac{\beta+2}{2},\frac{\gamma+2}{2}\right)$$

$$\text{DR's of PC}=\frac{\alpha}{2}:\frac{\beta-2}{2}:\frac{\gamma}{2}$$

$$\text{line L}\,||\,\text{PC}\Rightarrow\frac{-\alpha}{2}=\frac{\beta-2}{2}=\frac{\gamma}{4}=K(\text{let})$$

α = – 2K

β = 2K + 2

γ = 4K

use in (1) $\Rightarrow K=\frac{1}{6}$

value of α + β + 6γ = 24K + 2 = 6

30. Let y = y(x) be the solution of the differential equation $(x + y + 2)^2\, dx = dy$, $y(0) = -2$. Let the maximum and minimum values of the function $y = y(x)$ in $\left[0, \frac{\pi}{3}\right]$ be α and β, respectively. If $(3\alpha + \pi)^2 + \beta^2 = \gamma + \delta\sqrt{3}, \gamma, \delta \in \mathbb{Z}$, then γ + δ equals …..

Ans. (31)

Sol. $\frac{dy}{dx} = (x + y + 2)^2$ ...(1), $y(0) = -2$

Let $x + y + 2 = v$

$1 + \frac{dy}{dx} = \frac{dv}{dx}$

from (1) $\frac{dv}{dx} = 1 + v^2$

$\int \frac{dv}{1+v^2} = \int dx$

$\tan^{-1}(v) = x + C$

$\tan^{-1}(x + y + 2) = x + C$

at $x = 0$ $y = -2$ $\Rightarrow C = 0$

$\Rightarrow \tan^{-1}(x + y + 2) = x$

$y = \tan x - x - 2$

$f(x) = \tan x - x - 2,\ x \in \left[0, \frac{\pi}{3}\right]$

$f'(x) = \sec^2 x - 1 > 0 \Rightarrow f(x) \uparrow$

$f_{min} = f(0) = -2 = \beta$

$f_{max} = f\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3} - 2 = \alpha$

now $(3\alpha + \pi)^2 + \beta^2 = \gamma + \delta\sqrt{3}$

$\Rightarrow (3\alpha + \pi)^2 + \beta^2 = (3\sqrt{3} - 6)^2 + 4$

$\gamma + \delta\sqrt{3} = 67 - 36\sqrt{3}$

$\Rightarrow \gamma = 67$ and $\delta = -36 \Rightarrow \gamma + \delta = 31$