FINAL JEE–MAIN EXAMINATION – APRIL, 2023

(Held On Wednesday 12th April, 2023) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The number of five digit numbers, greater than 40000 and divisible by 5, which can be formed using the digits 0, 1, 3, 5, 7 and 9 without repetition, is equal to

(1) 120

(2) 132

(3) 72

(4) 96

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

5 x x x 0

7 x x x 0

7 x x x 5

9 x x x 0

9 x x x 5

So Required numbers = $5 \times {}^4P_3 = 120$

2. Let α, β be the roots of the quadratic equation $x^2 + \sqrt{6}x + 3 = 0$. Then $\dfrac{\alpha^{23}+\beta^{23}+\alpha^{14}+\beta^{14}}{\alpha^{15}+\beta^{15}+\alpha^{10}+\beta^{10}}$ is equal to

(1) 729

(2) 72

(3) 81

(4) 9

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\alpha, \beta = \dfrac{-\sqrt{6} \pm \sqrt{6-12}}{2} = \dfrac{-\sqrt{6} \pm \sqrt{6}\, i}{2}$

$= \sqrt{3} e^{\pm \frac{3\pi i}{4}}$

Required expression

$$= \frac{\left(\sqrt{3}\right)^{23}\left(2\cos\frac{69\pi}{4}\right) + \left(\sqrt{3}\right)^{14}\left(2\cos\frac{42\pi}{4}\right)}{\left(\sqrt{3}\right)^{15}\left(2\cos\frac{45\pi}{4}\right) + \left(\sqrt{3}\right)^{10}\left(2\cos\frac{30\pi}{4}\right)}$$

$\left(\sqrt{3}\right)^8 = 81$

3. Let $< a_n >$ be a sequence such that $a_1 + a_2 + ... + a_n = \dfrac{n^2+3n}{(n+1)(n+2)}$. If $28\sum_{k=1}^{10} \dfrac{1}{a_k} = p_1 p_2 p_3 .... p_m$, where $p_1, p_2, ...p_m$ are the first m prime numbers, then m is equal to

(1) 7

(2) 6

(3) 5

(4) 8

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $a_n = S_n - S_{n-1} = \dfrac{n^2+3n}{(n+1)(n+2)} - \dfrac{(n-1)(n+2)}{n(n+1)}$

$\Rightarrow a_n = \dfrac{4}{n(n+1)(n+2)}$

$\Rightarrow 28\sum_{k=1}^{10} \dfrac{1}{a_k} = 28\sum_{k=1}^{10} \dfrac{k(k+1)(k+2)}{4}$

$= \dfrac{7}{4}\sum_{k=1}^{10}\left(k(k+1)(k+2)(k+3) - (k-1)k(k+1)(k+2)\right)$

$= \dfrac{7}{4}.10.11.12.13 = 2.3.5.7.11.13$

So m = 6

4. Let the lines $l_1 : \dfrac{x+5}{3} = \dfrac{y+4}{1} = \dfrac{z-\alpha}{-2}$ and $l_2 : 3x + 2y + z - 2 = 0 = x - 3y + 2z - 13$ be coplanar. If the point P(a, b, c) on l_1 is nearest to the point Q(-4, -3, 2), then |a| + |b|+|c| is equal to

(1) 12

(2) 14

(3) 10

(4) 8

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $(3x + 2y + z - 2) + \mu(x - 3y + 2z - 13) = 0$

$3(3 + \mu) + 1.(2 - 3\mu) - 2(1 + 2\mu) = 0$

$9 - 4\mu = 0$

$\mu = \frac{9}{4}$

$4(-15 - 8 + \alpha - 2) + 9(-5 + 12 + 2\alpha - 13) = 0$

$-100 + 4\alpha - 54 + 18\alpha = 0$

$\Rightarrow \alpha = 7$

Let $P \equiv (3\lambda - 5, \lambda - 4, -2\lambda + 7)$

Direction ratio of PQ $(3\lambda - 1, \lambda - 1, -2\lambda + 5)$

But $PQ \perp \ell_1$

$\Rightarrow 3(3\lambda - 1) + 1.(\lambda - 1) - 2(-2\lambda + 5) = 0$

$\Rightarrow \lambda = 1$

$P(-2, -3, 5) \Rightarrow |a| + |b| + |c| = 10$

5. Let $P\left(\frac{2\sqrt{3}}{\sqrt{7}}, \frac{6}{\sqrt{7}}\right)$, Q, R and S be four points on the ellipse $9x^2 + 4y^2 = 36$. Let PQ and RS be mutually perpendicular and pass through the origin. If $\frac{1}{(PQ)^2} + \frac{1}{(RS)^2} = \frac{p}{q}$, where p and q are coprime, then p + q is equal to

(1) 143 (2) 137

(3) 157 (4) 147

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Let $R(2\cos\theta, 3\sin\theta)$

as $OP \perp OR$

so $\frac{3\sin\theta}{2\cos\theta} \times \frac{\frac{6}{\sqrt{7}}}{\frac{2\sqrt{3}}{\sqrt{7}}} = -1$

$\Rightarrow \tan\theta = \frac{-2}{3\sqrt{3}}$

$\Rightarrow R\left(\frac{-6\sqrt{3}}{\sqrt{31}}, \frac{6}{\sqrt{31}}\right)$ or $R\left(\frac{6\sqrt{3}}{\sqrt{31}}, \frac{-6}{\sqrt{31}}\right)$

Now $= \frac{1}{(PQ)^2} + \frac{1}{(RS)^2} = \frac{1}{4}\left(\frac{1}{(OP)^2} + \frac{1}{(OR)^2}\right)$

$= \frac{1}{4}\left(\frac{1}{\frac{48}{7}} + \frac{1}{\frac{144}{31}}\right) = \frac{1}{4}\left(\frac{7}{48} + \frac{31}{144}\right)$

$= \frac{13}{144}$

$\Rightarrow p + q = 157$

6. Let a, b, c be three distinct real numbers, none equal to one. If the vectors $a\hat{i} + \hat{j} + \hat{k}, \hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ are coplanar, then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is equal to

(1) 1

(2) – 1

(3) – 2

(4) 2

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$

$a(b-1)(c-1) - (1-a)(c-1) + (1-a)(1-b) = 0$

$a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$

$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$

$\Rightarrow -1 + \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$

$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$

7. If the local maximum value of the function $f(x) = \left(\frac{\sqrt{3e}}{2\sin x}\right)^{\sin^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$, is $\frac{k}{e}$, then $\left(\frac{k}{e}\right)^8 + \frac{k^8}{e^5} + k^8$ is equal to

(1) $e^5 + e^6 + e^{11}$

(2) $e^3 + e^5 + e^{11}$

(3) $e^3 + e^6 + e^{11}$

(4) $e^3 + e^6 + e^{10}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. **Let** $y = \left(\frac{\sqrt{3e}}{2\sin x}\right)^{\sin^2 x}$

$$\ln y = \sin^2 x . \ln\left(\frac{\sqrt{3e}}{2\sin x}\right)$$

$$\frac{1}{y}y' = \ln\left(\frac{\sqrt{3e}}{2\sin x}\right)2\sin x\cos x + \sin^2 x\frac{2\sin x}{\sqrt{3e}}\frac{\sqrt{3e}}{2}(-\text{cosec}\, x\cot x)$$

$$\frac{dy}{dx} = 0 \Rightarrow \ln\left(\frac{\sqrt{3e}}{2\sin x}\right)2\sin x\cos x - \sin x\cos x = 0$$

$$\Rightarrow \sin x\cos x\left[2\ln\left(\frac{\sqrt{3e}}{2\sin x}\right) - 1\right] = 0$$

$$\Rightarrow \ln\left(\frac{3e}{4\sin^2 x}\right) = 1 \Rightarrow \frac{3e}{4\sin^2 x} = e \Rightarrow \sin^2 x = \frac{3}{4}$$

$$\Rightarrow \sin x = \frac{\sqrt{3}}{2} \quad \left(\text{as } x \in \left(0, \frac{\pi}{2}\right)\right)$$

$$\Rightarrow \text{ local max value} = \left(\frac{\sqrt{3e}}{\sqrt{3}}\right)^{3/4} = e^{3/8} = \frac{k}{e}$$

$$\Rightarrow k^8 = e^{11}$$

$$\Rightarrow \left(\frac{k}{e}\right)^8 + \frac{k^8}{e^5} + k^8 = e^3 + e^6 + e^{11}$$

8. Let D be the domain of the function $f(x) = \sin^{-1}\left(\log_{3x}\left(\frac{6+2\log_3 x}{-5x}\right)\right)$. If the range of the function $g : D \rightarrow R$ defined by $g(x) = x - [x]$, ([x] is the greatest integer function), is (α, β), then $\alpha^2 + \frac{5}{\beta}$ is equal to

(1) 46

(2) 135

(3) 136

(4) 45

Official Ans. by NTA (2)

Allen Ans. (Bonus)

Sol. $\frac{6+2\log_3 x}{-5x} > 0 \;\&\; x > 0 \;\&\; x \neq \frac{1}{3}$

this gives $x \in \left(0, \frac{1}{27}\right)$ ... (1)

$$-1 \leq \log_{3x}\left(\frac{6+2\log_3 x}{-5x}\right) \leq 1$$

$$3x \leq \frac{6+2\log_3 x}{-5x} \leq \frac{1}{3x}$$

$15x^2 + 6 + 2\log_3 x \geq 0$ $\qquad$ $6 + 2\log_3 x + \frac{5}{3} \geq 0$

$x \in \left(0, \frac{1}{27}\right)$ ... (2) $\qquad$ $x \geq 3^{-\frac{23}{6}}$ ... (3)

from (1), (2) & (3)

$$x \in \left[3^{-\frac{23}{6}}, \frac{1}{27}\right)$$

$\therefore$ α is small positive quantity

& $\beta = \frac{1}{27}$

$\therefore$ $\alpha^2 + \frac{5}{\beta}$ is just greater than 135

Ans. (Bonus)

9. Let $y = y(x)$, $y > 0$, be a solution curve of the differential equation $(1 + x^2)\, dy = y\,(x - y)\, dx$. If $y(0) = 1$ and $y(2\sqrt{2}) = \beta$, then

(1) $e^{3\beta^{-1}} = e(3+2\sqrt{2})$

(2) $e^{\beta^{-1}} = e^{-2}\left(5+\sqrt{2}\right)$

(3) $e^{\beta^{-1}} = e^{-2}\left(3+2\sqrt{2}\right)$

(4) $e^{3\beta^{-1}} = e\left(5+\sqrt{2}\right)$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $(1+x^2)\, dy = y\,(x - y)dx$

$y(0) = 1.\; y(2\sqrt{2}) = \beta$

$$\frac{dy}{dx} = \frac{yx - y^2}{1+x^2}$$

$$\frac{dy}{dx} + y\left(\frac{-x}{1+x^2}\right) = \left(\frac{-1}{1+x^2}\right)y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\left(\frac{-x}{1+x^2}\right) = \frac{-1}{1+x^2}$$

put $\frac{1}{y} = t$ then $\frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx}+t\frac{x}{1+x^2}=\frac{1}{1+x^2}$$

$$\text{I.F} = e^{\int\frac{x}{1+x^2}dx}=e^{\frac{1}{2}\ln(1+x^2)} = \sqrt{1+x^2}$$

$$t\sqrt{1+x^2}=\int\frac{1}{\sqrt{1+x^2}}dx$$

$$\frac{\sqrt{1+x^2}}{y}=\ln\left(x+\sqrt{x^2+1}\right)+c$$

$y(0) = 1 \quad \Rightarrow c=1$

$$\Rightarrow \sqrt{1+x^2}=y\ln(e(x+\sqrt{x^2+1}))$$

$$\beta=\frac{3}{\ln(e(3+2\sqrt{2}))} \Rightarrow \frac{3}{\beta}=\ln(e(3+2\sqrt{2}))$$

$$e^{\frac{3}{\beta}}=e(3+2\sqrt{2})$$

10. Among the two statements

(S1) : $(p\Rightarrow q)\wedge(q\wedge(\sim q))$ is a contradiction and

(S2) : $(p\wedge q)\vee((\sim p)\wedge q)\vee$

$(p\wedge(\sim q))\vee((\sim p)\wedge(\sim q))$ is a tautology

(1) only (S2) is true

(2) only (S1) is true

(3) both are false.

(4) both are true

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $S_1:(p\to q)\wedge(p\wedge(\sim q))$

p	q	$p\to q$	$p\wedge(\sim q)$	S1
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

$\Rightarrow$ S_1 is Contradiction

S_2

p	q	$p\wedge q$	$(\sim p\wedge q)$	$(p\wedge\sim q)$	$(\sim p)\wedge(\sim q)$	S_2
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	F	F	F	T	T

S_2 is tautology

11. Let $\lambda\in Z,\vec{a}=\lambda\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$. Let $\vec{c}$ be a vector such that $(\vec{a}+\vec{b}+\vec{c})\times\vec{c}=\vec{0},\vec{a}.\vec{c}=-17$ and $\vec{b}.\vec{c}=-20$. Then $|\vec{c}\times(\lambda\hat{i}+\hat{j}+\hat{k})|^2$ is equal to

(1) 62

(2) 46

(3) 53

(4) 49

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $(\vec{a}+\vec{b}+\vec{c})\times\vec{c}=0$

$(\vec{a}+\vec{b})\times\vec{c}=0$

$\vec{c}=\alpha(\vec{a}+\vec{b})=\alpha(\lambda+3)\hat{i}+\alpha\hat{k}$

$\vec{b}.\vec{c}=-20\Rightarrow 3\alpha(\lambda+3)+2\alpha=-20$

$\vec{a}.\vec{c}=-17\Rightarrow \alpha\lambda(\lambda+3)-\alpha=-17$

$\Rightarrow \alpha(3\lambda+9+2)=-20$

$\alpha(\lambda^2+3\lambda-1)=-17$

$17(3\lambda+11)=20(\lambda^2+3\lambda-1)$

$20\lambda^2+9\lambda-207=0$

$\lambda=3 \qquad (\lambda\in Z)$

$\Rightarrow\alpha=-1 \qquad \Rightarrow\vec{c}=-(6\hat{i}+\hat{k})$

$\vec{v}=\vec{c}\times(3\hat{i}+\hat{j}+\hat{k})$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -6 & 0 & -1\\ 3 & 1 & 1\end{vmatrix}=\hat{i}+3\hat{j}-6\hat{k}$$

$|\vec{v}|^2=(-1)^2+3^2+6^2=46$

12. The sum, of the coefficients of the first 50 terms in the binomial expansion of $(1-x)^{100}$, is equal to

(1) $-{}^{101}C_{50}$

(2) ${}^{99}C_{49}$

(3) $-{}^{99}C_{49}$

(4) ${}^{101}C_{50}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $(1-x)^{100} = C_0 - C_1x + C_2x^2 - C_3x^3 + \ldots C_{99}x^{99} + C_{100}x^{100}$

$$\Rightarrow C_0 - C_1 + C_2 - C_3 + \ldots\ldots - C_{99} + C_{100} = 0$$

$$2(C_0 - C_1 + C_2 + \ldots\ldots - C_9) + C_{50} = 0$$

$$C_0 - C_1 + C_2 + \ldots C_{99} = -\frac{1}{2}\,{}^{100}C_{50}$$

$$-\frac{1}{2}\frac{100!}{50!50!} = -\frac{1}{2}\times\frac{100\times 99!}{50!50!} = -{}^{99}C_{49}$$

13. The area of the region enclosed by the curve $y = x^3$ and its tangent at the point (-1, -1) is

(1) $\frac{27}{4}$

(2) $\frac{19}{4}$

(3) $\frac{23}{4}$

(4) $\frac{31}{4}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. equation of tangent : $y + 1 = 3(x + 1)$

i.e. $y = 3x + 2$

Point of intersection with curve (2, 8)

So Area = $\int_{-1}^{2}((3x+2) - x^3)dx = \frac{27}{4}$

14. Let $A = \begin{bmatrix} 1 & \frac{1}{51} \\ 0 & 1 \end{bmatrix}$. If $B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} A \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$, then the sum of all the elements of the matrix $\sum_{n=1}^{50} B^n$ is equal to

(1) 100

(2) 50

(3) 75

(4) 125

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Let $C = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$$DC = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

B = CAD

$$B^n = \underbrace{(CAD)(CAD)(CAD)\ldots(CAD)}_{n-\text{times}}$$

$\Rightarrow B^n = CA^nD$ ....(1)

$$A^2 = \begin{bmatrix} 1 & \frac{1}{51} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{51} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{51} \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & \frac{3}{51} \\ 0 & 1 \end{bmatrix}$$

similarly $A^n = \begin{bmatrix} 1 & \frac{n}{51} \\ 0 & 1 \end{bmatrix}$

$$B^n = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \frac{n}{51} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{n}{51}+2 \\ -1 & -\frac{n}{51}-1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{n}{51}+1 & \frac{n}{51} \\ -\frac{n}{51} & 1-\frac{n}{51} \end{bmatrix}$$

$$\sum_{n=1}^{50} B^n = \begin{bmatrix} 25+50 & 25 \\ -25 & -25+50 \end{bmatrix} = \begin{bmatrix} 75 & 25 \\ -25 & 25 \end{bmatrix}$$

Sum of the elements = 100

15. Let the plane P : $4x - y + z = 10$ be rotated by an angle $\frac{\pi}{2}$ about its line of intersection with the plane $x + y - z = 4$. If α is the distance of the point (2, 3, – 4) from the new position of the plane P, then 35α is

(1) 90
(2) 85
(3) 105
(4) 126

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Let equation in new position is

$(4x - y + z - 10) + \lambda(x + y - z - 4) = 0$

$4(4+\lambda) - 1.(-1+\lambda) + 1.(1-\lambda) = 0$

$\Rightarrow \lambda = -9$

So equation in new position is

$-5x - 10y + 10z + 26 = 0$

$\Rightarrow \alpha = \frac{54}{15}$

16. If $\frac{1}{n+1}\ {}^nC_n + \frac{1}{n}\ {}^nC_{n-1}$

$+ ... + \frac{1}{2}\ {}^nC_1 + {}^nC_0 = \frac{1023}{10}$ then n is equal to

(1) 6

(2) 9

(3) 8

(4) 7

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\sum_{r=0}^{n} \frac{{}^nC_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} {}^{n+1}C_{r+1}$

$= \frac{1}{n+1}(2^{n+1} - 1) = \frac{1023}{10}$

$n + 1 = 10 \Rightarrow n = 9$

17. Let C be the circle in the complex plane with centre $z_0 = \frac{1}{2}(1+3i)$ and radius r = 1. Let z_1 = 1+ i and the complex number z_2 be outside the circle C such that $|z_1 - z_0|$ $|z_2 - z_0| = 1$. If z_0, z_1 and z_2 are collinear, then the smaller value of $|z_2|^2$ is equal to

(1) $\frac{13}{2}$

(2) $\frac{5}{2}$

(3) $\frac{3}{2}$

(4) $\frac{7}{2}$

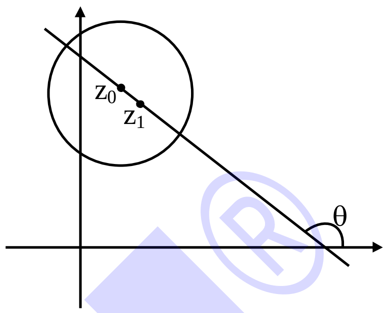
Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $|z_1 - z_0| = \left|\frac{1-i}{2}\right| = \frac{1}{\sqrt{2}}$

$\Rightarrow |z_2 - z_o| = \sqrt{2}$; centre $\left(\frac{1}{2}, \frac{3}{2}\right)$

$z_o\left(\frac{1}{2}, \frac{3}{2}\right)$ and z_1 (1, 1)



$\tan\theta = -1 \Rightarrow \theta = 135°$

$z_2\left(\frac{1}{2} + \sqrt{2}\cos 135°, \frac{3}{2} + \sqrt{2}\sin 135°\right)$

or

$\left(\frac{1}{2} - \sqrt{2}\cos 135°, \frac{3}{2} - \sqrt{2}\sin 135°\right)$

$\Rightarrow z_2\left(-\frac{1}{2}, \frac{5}{2}\right)$ or $z_2\left(\frac{3}{2}, \frac{1}{2}\right)$

$\Rightarrow |z_2|^2 = \frac{26}{4}, \frac{5}{2}$

$\Rightarrow |z_2|^2_{min} = \frac{5}{2}$

18. If the point $\left(\alpha, \frac{7\sqrt{3}}{3}\right)$ lies on the curve traced by the mid-points of the line segments of the lines x cos θ + y sin θ = 7, $\theta \in \left(0, \frac{\pi}{2}\right)$ between the co-ordinates axes, then α is equal to

(1) 7

(2) –7

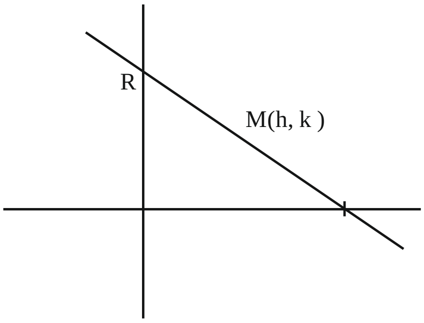
(3) $-7\sqrt{3}$

(4) $7\sqrt{3}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $pt(\alpha, \frac{7\sqrt{3}}{3})$



$x\cos\theta + y\sin\theta = 7$

$x\text{ - intercept } = \frac{7}{\cos\theta}$

$y\text{ - intercept } = \frac{7}{\sin\theta}$

$A:\left(\frac{7}{\cos\theta},0\right)\quad B:\left(0,\frac{7}{\sin\theta}\right)$

Locus of mid pt M : (h, k)

$h = \frac{7}{2\cos\theta}, k = \frac{7}{2\sin\theta}$

$\frac{7}{2\sin\theta} = \frac{7\sqrt{3}}{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\alpha = \frac{7}{2\cos\theta} = 7$

19. Two dice A and B are rolled, Let the numbers obtained on A and B be α and β respectively. If the variance of $\alpha - \beta$ is $\frac{p}{q}$, where p and q are co-prime, then the sum of the positive divisors of p is equal to

(1) 36

(2) 48

(3) 31

(4) 72

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

α-β	Case	P
5	(6, 1)	1/36
4	(6, 2) (5, 1)	2/36
3	(6, 3) (5, 2) (4, 1)	3/36
2	(6, 4) (5, 3) (4, 3) (3, 1)	4/36
1	(6, 5) (5, 4) (4, 3) (3, 2) (2, 1)	5/36
0	(6, 6) (5, 5) …… (1, 1)	6/36
-1	-----	5/36
-2	-----	4/36
-3	-----	3/36
-4	(2, 6) (1, 5)	2/36
-5	(1, 6)	1/36

$$\sum(x^2) = \sum x^2P(x) = 2\left[\frac{25}{36}+\frac{32}{36}+\frac{27}{36}+\frac{16}{36}+\frac{5}{36}\right]$$

$$= \frac{105}{18} = \frac{35}{6}$$

$\mu = \sum(x) = 0$ as data is symmetric

$\sigma^2 = \sum(x^2) = \sum x^2P(x) = \frac{35}{6}\quad P = 35 = 5\times 7$

Sum of divisors = $(5^0+5^1)(7^0+7^1) = 6\times 8 = 48$

20. In a triangle ABC, if cos A + 2 cos B + cos C = 2 and the lengths of the sides opposite to the angles A and C are 3 and 7 respectively, then cos A – cos C is equal to

(1) $\frac{3}{7}$

(2) $\frac{9}{7}$

(3) $\frac{10}{7}$

(4) $\frac{5}{7}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\cos A + \cos C = 2(1 - \cos B)$

$$2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = 4\sin^2 B/2$$

as $\cos\left(\frac{A+C}{2}\right) = \sin\frac{B}{2}$

so $\cos\frac{A-C}{2} = 2\sin\frac{B}{2}$

2cos B/2 cos $\frac{A-C}{2}$ = 4sin B/2 cos B/2

$$2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right) = 4\sin B/2\cos B/2$$

Sin A + sin C = 2 sin B

a + c = 2b $\Rightarrow$ a = 3, c = 7, b = 5

$$\cos A - \cos C = \frac{b^2+c^2-a^2}{2bc} - \frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{25+49-9}{70} - \frac{9+25-49}{30}$$

$$= \frac{65}{70} + \frac{1}{2} = \frac{20}{14} = \frac{10}{7}$$

SECTION-B

21. A fair n (n > 1) faces die is rolled repeatedly until a number less than n appears. If the mean of the number of tosses required is $\frac{n}{9}$, then n is equal to __________.

Official Ans. by NTA (10.00)

Allen Ans. (10.00)

Sol. Mean $= 1.\frac{n-1}{n} + 2\frac{1}{n}\left(\frac{n-1}{n}\right) + 3\left(\frac{1}{n}\right)^2\left(\frac{n-1}{n}\right)$ ...

$$\frac{n}{9} = \left(\frac{n-1}{n}\right)\left(1 + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right)^2 ......\right)$$

$$\frac{n}{9} = \left(\frac{n-1}{n}\right)\left(1 - \frac{1}{n}\right)^{-2} = \left(\frac{n-1}{n}\right)\cdot\frac{n^2}{(n-1)^2}$$

$$\frac{n}{9} = \frac{n}{n-1} \Rightarrow n = 10$$

22. Let the digits a, b, c be in A.P. Nine-digit numbers are to be formed using each of these three digits thrice such that three consecutive digits are in A.P. at least once. How many such numbers can be formed?

Official Ans. by NTA (1260)

Allen Ans. (1260)

Sol. abc or cba

$$\_\ \_\ \_\ \underset{c\ b\ a}{\overset{a\ b\ c}{\_\ \_\ \_}}\ \_\ \_\ \_$$

$$\frac{{}^7C_1 \times 2 \times 6!}{2!2!2!} = 1260$$

23. Let [x] be the greatest integer $\le$ x. Then the number of points in the interval (–2,1), where the function $f(x) = |[x]| + \sqrt{x - [x]}$ is discontinuous is________.

Official Ans. by NTA (2.00)

Allen Ans. (2.00)

Sol. Need to check at doubtful points

discont at $x \in I$ only

at x = – 1 $\Rightarrow f(-1^+) = 1 + 0 = 1$

$\Rightarrow f(-1^-) = 2 + 1 = 3$

at x = 0 $\Rightarrow f(0^+) = 0 + 0 = 0$

$\Rightarrow f(0^-) = 1 + 1 = 2$

at x = 1 $\Rightarrow f(1^+) = 1 + 0 = 1$

$\Rightarrow f(1^-) = 0 + 1 = 1$

discont. at two points

24. Let the plane x + 3y – 2z + 6 = 0 meet the co-ordinate axes at the points A,B,C. If the orthocentre of the triangle ABC is $\left(\alpha, \beta, \frac{6}{7}\right)$, then $98(\alpha + \beta)^2$ is equal to______.

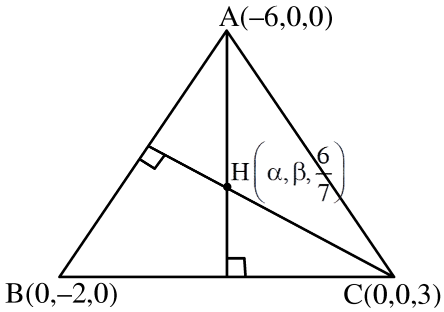
Official Ans. by NTA (288.00)

Allen Ans. (288.00)

Sol. A (– 6, 0, 0) B (0, – 2, 0) C = (0, 0, 3)

$\overrightarrow{AB} = 6\hat{i} - 2\hat{j}, \quad \overrightarrow{BC} = 2\hat{j} + 3\hat{k},$

$\overrightarrow{AC} = 6\hat{i} + 3\hat{k}$



$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$$

$$\left(\alpha + 6, \beta, \frac{6}{7}\right) \cdot (0, 2, 3) = 0$$

$$\boxed{\beta = \frac{-9}{7}}$$

$$\overrightarrow{CH} \cdot \overrightarrow{AB} = 0$$

$$\left(\alpha, \beta, \frac{-15}{7}\right) \cdot (6, -2, 0) = 0$$

$$6\alpha - 2\beta = 0$$

$$\alpha = \frac{-3}{7}$$

$$98(\alpha + \beta)^2 = (98)\frac{(144)}{49} = 288$$

25. Let $I(x) = \int \sqrt{\frac{x+7}{x}}\, dx$ and $I(9) = 12 + 7 \log_e 7$. If $I(1) = \alpha + 7 \log_e (1 + 2\sqrt{2})$, then α^4 is equal to___________.

Official Ans. by NTA (64.00)

Allen Ans. (64.00)

Sol. $\int \sqrt{\frac{x+7}{x}}\, dx$

Put $x = t^2$

$dx = 2tdt$

$$\int 2\sqrt{t^2 + 7}\, dt = 2\int \sqrt{t^2 + \sqrt{7}^2}\, dt$$

$$I(t) = 2\left[\frac{t}{2}\sqrt{t^2 + 7} + \frac{7}{2}\ln\left|t + \sqrt{t^2 + 7}\right|\right] + C$$

$$I(x) = \sqrt{x}\sqrt{x+7} + 7\, ln\left|\sqrt{x} + \sqrt{x+7}\right| + C$$

$$I(9) = 12 + 7\, ln7 = 12 + 7\,(ln\,(3 + 4)) + C$$

$$\Rightarrow C = 0$$

$$I(x) = \sqrt{x}\sqrt{x+7} + 7\, ln\left(\sqrt{x} + \sqrt{x+7}\right)$$

$$I(1) = 1\sqrt{8} + 7\, ln\left(1 + \sqrt{8}\right)$$

$$I(1) = \sqrt{8} + 7\, ln\,(1 + 2\sqrt{2})$$

$$\alpha = \sqrt{8}$$

$$\alpha^4 = (8^{1/2})^4$$

$$\alpha^4 = 8^2 = 64$$

26. Let $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$. If $\sum_{k=1}^{n} D_k = 96$, then n is equal to

Official Ans. by NTA (6.00)

Allen Ans. (6.00)

Sol. $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$

$$\sum_{k=1}^{n} D_k = 96 \Rightarrow$$

$$\begin{vmatrix} \sum_{k=1}^{n} 1 & \Sigma 2k & \Sigma(2k-1) \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix} = 96$$

$$\Rightarrow \begin{vmatrix} n & n^2+n & n^2 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix} = 96$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\begin{vmatrix} n & n^2+n & n^2 \\ 0 & 2 & 0 \\ 0 & 0 & n+2 \end{vmatrix} = 96$$

$$\Rightarrow n\,(2n + 4) = 96 \Rightarrow n\,(n + 2) = 48 \Rightarrow n = 6$$

27. Let the positive numbers a_1, a_2, a_3, a_4 and a_5 be in a G.P. Let their mean and variance be $\frac{31}{10}$ and $\frac{m}{n}$ respectively, where m and n are co-prime. If the mean of their reciprocals is $\frac{31}{40}$ and $a_3 + a_4 + a_5 = 14$, then m + n is equal to_________.

Official Ans. by NTA (211)

Allen Ans. (211)

Sol. Let $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Given $\frac{a}{r^2} + \frac{a}{r} + a + ar + ar^2 = 5 \times \frac{31}{10}$ ...(1)

And $\frac{r^2}{a} + \frac{r}{a} + \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} = 5 \times \frac{31}{40}$ ...(2)

(1) ÷ (2) $a^2 = 4 \Rightarrow a = 2 \quad \therefore r + \frac{1}{r} = 5/2 \quad (a \neq -2)$

$\Rightarrow r = 2$

$\therefore$ Now $\frac{1}{2}$, 1, 2. 4, 8

$\therefore \sigma^2 = \frac{\Sigma x^2}{N} - \left(\frac{\Sigma x}{N}\right)^2$

$= \frac{186}{25} = \frac{M}{N} \Rightarrow 211 = m + n$

28. The number of relations, on the set {1,2,3} containing (1,2) and (2,3), which are reflexive and transitive but not symmetric, is______

Official Ans. by NTA (4.00)

Allen Ans. (3.00)

Sol. A = {1,2, 3}

For Reflexive (1, 1) (2, 2), (3, 3) $\in$ R

For transitive : (1, 2) and (2, 3) $\in R \Rightarrow$ (1, 3) $\in$ R

Not symmetric : (2, 1) and (3, 2) $\notin$ R

R_1 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}

R_2 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3) (2, 1)}

R_3 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3) (3, 2)}

29. If $\int_{-0.15}^{0.15} |100x^2 - 1| dx = \frac{k}{3000}$, then k is equal to______.

Official Ans. by NTA (575)

Allen Ans. (575)

Sol. $\int_{-0.15}^{0.15} |100x^2 - 1| dx = 2\int_{0}^{0.15} |100x^2 - 1| dx$

Now $100x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{100} \Rightarrow x = 0.1$

$I = 2\left[\int_0^{0.1} (1 - 100x^2) dx + \int_{0.1}^{0.15} (100x^2 - 1) dx\right]$

$I = 2\left[x - \frac{100}{3}x^3\right]_0^{0.1} + 2\left[\frac{100x^3}{3} - x\right]_{0.1}^{0.15}$

$= 2\left[0.1 - \frac{0.1}{3}\right] + 2\left[\frac{0.3375}{3} - 0.15 - \frac{0.1}{3} + 0.1\right]$

$= 2\left[0.2 - \frac{0.2}{3} + 0.1125 - 0.15\right]$

$= 2\left[\frac{5}{100} - \frac{2}{30} + \frac{1125}{10000}\right] = 2\left(\frac{1500 - 2000 + 3375}{30000}\right)$

$= \frac{575}{3000} \Rightarrow k = 575$

30. Two circles in the first quadrant of radii r_1 and r_2 touch the coordinate axes. Each of them cuts off an intercept of 2 units with the line x + y = 2. Then $r_1^2 + r_2^2 - r_1r_2$ is equal to________.

Official Ans. by NTA (7.00)

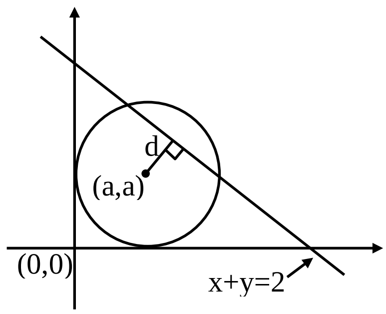
Allen Ans. (7.00)

Sol. Circle $(x - a)^2 + (y - a)^2 = a^2$

$x^2 + y^2 - 2ax - 2ay + a^2 = 0$

intercept = 2

$\Rightarrow 2\sqrt{a^2 - d^2} = 2$



Where d = perpendicular distance of centre from line x + y = 2

$\Rightarrow 2\sqrt{a^2 - \left(\frac{a + a - 2}{\sqrt{2}}\right)^2} = 2$

$\Rightarrow a^2 - \frac{(2a-2)^2}{2} = 1 \Rightarrow 2a^2 - 4a^2 + 8a - 4 = 2$

$\Rightarrow 2a^2 - 8a + 6 = 0 \Rightarrow a^2 - 4a + 3 = 0$

$\therefore r_1 + r_2 = 4$ and $r_1r_2 = 3$

$\therefore r_1^2 + r_2^2 - r_1r_2 = (r_1 + r_2)^2 - 3r_1r_2$

= 16 – 9 = 7