FINAL JEE–MAIN EXAMINATION – APRIL, 2023

(Held On Thursday 06th April, 2023) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS **TEST PAPER WITH SOLUTION**

SECTION-A

1. Three dice are rolled. If the probability of getting different numbers on the three dice is $\frac{p}{q}$, where p and q are co-prime, then q – p is equal to

(1) 4
(2) 3
(3) 1
(4) 2

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. Total number of ways = 6^3 = 216

Favourable outcomes $^6p_3$ = 120

$\Rightarrow \text{Probability} = \frac{120}{216} = \frac{5}{9}$

$\Rightarrow p = 5, q = 9$

$\Rightarrow q - p = 4$

2. Among the statements:

(S1) : $2023^{2022} - 1999^{2022}$ is divisible by 8.

(S2) : $13(13)^n - 11n - 13$ is divisible by 144 for infinitely many $n \in N$.

(1) both (S1) and (S2) are incorrect
(2) only (S2) is correct
(3) both (S1) and (S2) are correct
(4) only (S1) is correct

Official Ans. by NTA (3)
Allen Ans. (3)

Sol. $S_1 = (1999+24)^{2022} - (1999)^{2022}$

$\Rightarrow {}^{2022}C_1(1999)^{2021}(24) + {}^{2022}C_2(1999)^{2020}(24)^2 + ...\text{so on}$

S_1 is divisible by 8

$S_2 : 13(13^n) - 11n - 13$

$13^n = (1+12)^n = 1 + 12n + {}^nC_2 12^2 + {}^nC_3 12^3......$

$13(13^n) - 11n - 13 = 145n + {}^nC_2\, 12^2 + {}^nC_3 12^3.......$

If ($n = 144m,\ m \in N$), then it is divisible by 144

For infinite value of n.

3. $\lim_{n\to\infty}\left\{\left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}} - 2^{\frac{1}{5}}\right).......\left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}}\right)\right\}$

is equal to

(1) $\frac{1}{\sqrt{2}}$
(2) 1
(3) $\sqrt{2}$
(4) 0

Official Ans. by NTA (4)
Allen Ans. (4)

Sol. $\left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)^n < \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}} - 2^{\frac{1}{5}}\right)\left(2^{\frac{1}{2}} - 2^{\frac{1}{7}}\right)$

$----\left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}}\right) < \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}}\right)^n$

$\left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)^n < L < \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}}\right)^n$

$\lim_{n\to\infty}\left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)^n = 0$ and $\lim_{n\to\infty}\left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}}\right)^n = 0$

$\Rightarrow \lim_{n\to\infty} L = 0$

4. Let $a \neq b$ be two non-zero real numbers.

Then the number of elements in the set

$X = \left\{z \in C : \text{Re}(az^2 + bz) = a \text{ and } \text{Re}(bz^2 + az) = b\right\}$

is equal to

(1) 1
(2) 3
(3) 0
(4) 2

Official Ans. by NTA (3)
Allen Ans. (Bonus)

Sol. $\text{Re}(az^2 + bz) = a$

$az^2 + bz + a\bar{z}^2 + b\bar{z} = 2a$

$a(z^2 + \bar{z}^2) + b(z + \bar{z}) = 2a \quad ....(1)$

$\text{Re}(bz^2 + az) = b$

$bz^2 + az + b\bar{z}^2 + a\bar{z} = 2b$

$b(z^2 + \bar{z}^2) + a(z + \bar{z}) = 2b \quad ....(2)$

(1) × b – (2) × (a)

$\Rightarrow \quad (b^2 - a^2)(z + \bar{z}) = 0$

$\Rightarrow \quad (z+\bar{z}) = 0 \quad (a^2 \neq b^2)$

(1) × a – (2) × (b)

$\Rightarrow \quad (a^2-b^2)(z+\bar{z}) = 2(a^2-b^2) \quad (a^2 \neq b^2)$

$z^2+\bar{z}^2 = 2$

$\Rightarrow \quad (z+\bar{z})^2 - 2z\bar{z} = 2$

$z\bar{z} = -1$

$\Rightarrow \quad 1 + 1^2 = -1$

$\Rightarrow$ No solution

But when a = – b,

$\text{Re}(az^2 - az) = a$

$\Rightarrow \quad \text{Re}\left(a(x^2-y^2+i2xy) - a(x+iy)\right) = a$

$\Rightarrow \quad a(x^2-y^2) - ax = a$

$\Rightarrow \quad x^2-y^2-x = 1$

$\Rightarrow \quad x^2-x-1 = y^2$

For any real values of y there two values of x, hence infinite complex numbers are possible.

5. Let the sets A and B denote the domain and range respectively of the function $f(x) = \dfrac{1}{\sqrt{\lceil x \rceil - x}}$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Then among the statements

(S1) : $A \cap B = (1, \infty) - N$ and

(S2) : $A \cup B = (1, \infty)$

(1) only (S1) is true

(2) both (S1) and (S2) are true

(3) neither (S1) nor (S2) is true

(4) only (S2) is true

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $f(x) = \dfrac{1}{\sqrt{\lceil x \rceil - x}}$

If $x \in I$ $\lceil x \rceil = [x]$ (greatest integer function)

If $x \notin I$ $\lceil x \rceil = [x] + 1$

$$\Rightarrow f(x) = \begin{cases} \dfrac{1}{\sqrt{[x]-x}}, & x \in I \\ \dfrac{1}{\sqrt{[x]+1-x}}, & x \notin I \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \dfrac{1}{\sqrt{-\{x\}}}, & x \in I, \text{ (does not exist)} \\ \dfrac{1}{\sqrt{1-\{x\}}}, & x \notin I \end{cases}$$

$\Rightarrow$ domain of f(x) = R – I

Now, $f(x) = \dfrac{1}{\sqrt{1-\{x\}}}, x \notin I$

$\Rightarrow 0 < \{x\} < 1$

$\Rightarrow 0 < \sqrt{1-\{x\}} < 1$

$\Rightarrow \dfrac{1}{\sqrt{1-\{x\}}} > 1$

$\Rightarrow$ Range $(1, \infty)$

$\Rightarrow A = R - I$

$B = (1, \infty)$

So, $A \cap B = (1, \infty) - N$

$A \cup B \neq (1, \infty)$

$\Rightarrow$ S1 is only correct

6. If the solution curve f(x, y) = 0 of the differential equation $(1+\log_e x)\dfrac{dx}{dy} - x\log_e x = e^y$, x > 0, passes through the points (1,0) and (α , 2) then α^α is equal to

(1) $e^{2e^{\sqrt{2}}}$

(2) $e^{\sqrt{2}e^2}$

(3) e^{e^2}

(4) e^{2e^2}

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $(1+\ln x)\dfrac{dx}{dy} - x\ln x = e^y$

Let x lnx = t

$(1+\ln x)\dfrac{dx}{dy} = \dfrac{dt}{dy}$

$\dfrac{dt}{dy} - t = e^y$

If $= e^{\int -dy} = e^{-y}$

$t.e^{-y} = \int e^y e^{-y} dy + c$

$te^{-y} = y + c$

$x\ln x\, e^{-y} = y + c$

$x\ln x = ye^y + ce^y$

(1, 0) $\boxed{0 = C}$

$\Rightarrow x\ln x = ye^y$

$\Rightarrow \alpha\ln\alpha = 2e^2$

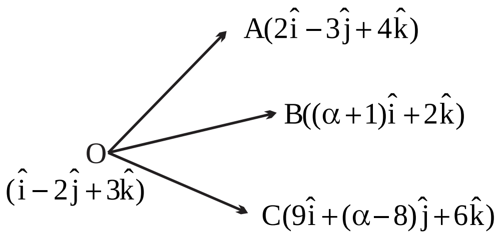
$\alpha^\alpha = e^{2e^2}$

7. The sum of all values of α, for which the points whose position vectors $\hat{i}-2\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{j}+4\hat{k}$, $(\alpha+1)\hat{i}+2\hat{k}$ and $9\hat{i}+(\alpha-8)\hat{j}+6\hat{k}$ are coplanar, is equal to

(1) 6
(2) 4
(3) –2
(4) 2

Official Ans. by NTA (4)
Allen Ans. (4)

Sol.



[OA OB OC] = 0

$$\begin{vmatrix} 1 & -1 & 1 \\ \alpha & 2 & -1 \\ 8 & \alpha-6 & 3 \end{vmatrix} = 0$$

$\Rightarrow \alpha^2 - 2\alpha - 8 = 0$

$\Rightarrow (\alpha-4)(\alpha+2) = 0$

$\therefore \alpha = 4, -2$

8. For the system of equations

$x + y + z = 6$

$x + 2y + \alpha z = 10$

$x + 3y + 5z = \beta$, which one of the following is <u>NOT</u> true?

(1) System has a unique solution for $\alpha = 3$, $\beta \neq 14$.
(2) System has no solution for $\alpha = 3$, $\beta = 24$.
(3) System has a unique solution for $\alpha = -3$, $\beta = 14$.
(4) System has infinitely many solutions for $\alpha = 3$, $\beta = 14$.

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $x + y + z$

$x + 2y + \alpha z = 10$

$x + 3y + 5z = \beta$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & \alpha \\ 1 & 3 & 5 \end{vmatrix} = 1(10-3\alpha) - 1(5-\alpha) + 1(3-z)$$

$= 10 - 3\alpha - 5 + \alpha + 1$

$= 6 - 2\alpha$

For unique solution $6 - 2\alpha \neq 0 \Rightarrow \alpha \neq 3$

9. The area bounded by the curves $y = |x-1| + |x-2|$ and $y = 3$ is equal to

(1) 3
(2) 4
(3) 5
(4) 6

Official Ans. by NTA (2)
Allen Ans. (2)

Sol. $y = |x-1| + |x-2|$ and $y = 3$

$\therefore$ Required area $= \frac{1}{2}(1+3)\times 2 = 4$

10. Let P be a square matrix such that $P^2 = I - P$. For $\alpha, \beta, \gamma, \delta \in N$, if $P^\alpha + P^\beta = \gamma I - 29P$ and $P^\alpha - P^\beta = \delta I - 13P$, then $\alpha + \beta + \gamma - \delta$ is equal to

(1) 18
(2) 40
(3) 24
(4) 22

Official Ans. by NTA (3)
Allen Ans. (3)

Sol. $P^2 = I - P$

$P^\alpha + P^\beta = \gamma I - 29P$, $P^\alpha - P^\beta = \delta I - 13P$

$P^4 = (I-P)^2 = I - 2P + P^2 = 2I - 3P$

$P^6 = (2I - 3P)(I - P) = 5I - 8P$

$P^8 = (2I - 3P)^2 = 4I - 12P + 9(I - P) = 13I - 21P$

$P^8 + P^6 = 18I - 29P$

$P^8 - P^6 = 8I - 13P$

$\alpha = 8$; $\beta = 6$; $\gamma = 18$, $\delta = 8$

$\alpha + \beta + \gamma - \delta = 8 + 6 + 18 - 8 = 24$

11. All the letters of the word PUBLIC are written in all possible orders and these words are written as in a dictionary with serial numbers. Then the serial number of the word PUBLIC is

(1) 580
(2) 582
(3) 578
(4) 576

Official Ans. by NTA (2)
Allen Ans. (2)

Sol. B ....→ 5! = 120

C ....→ 5! = 120

I ....→ 5! = 120

L ....→ 5! = 120

PB ....→ 4! = 24

PC ....→ 4! = 24

PL ....→ 4! = 24

PI ....→ 4! = 24

$P \cup BC$ ....→ 2! = 2

$P \cup BI$ ………→ 2! = 2

$P \cup BLC$ ……..→ 1! = 1

$P \cup BLIC$ …….→ = 1

Serial number = 4(120) + 4(24) + 6 = 582

12. Let the line L pass through the point (0, 1, 2), intersect the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and be parallel to the plane 2x + y – 3z = 4. Then the distance of the point P(l, –9, 2) from the line L is

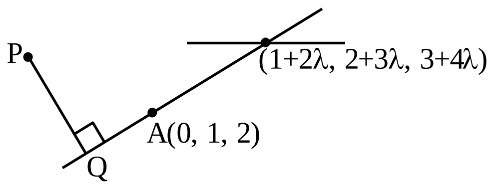
(1) 9 (2) $\sqrt{54}$

(3) $\sqrt{69}$ (4) $\sqrt{74}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



$\overrightarrow{AB}\cdot\vec{n}$

$\Rightarrow \left[(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1+4\lambda)\hat{k}\right]\cdot\left(2\hat{i}+\hat{j}-3\hat{k}\right)$

$2 + 4\lambda + 1 + 3\lambda - 3 - 12\lambda = 0$

$5\lambda = 0 \Rightarrow \lambda = 0$

Line $\overrightarrow{AB}$, $\vec{r}=\hat{j}+2\hat{k}+\mu\left(\hat{i}+\hat{j}+\hat{k}\right)$

General form: $Q(\mu, 1+\mu, 2+\mu)$

$\therefore \overrightarrow{PQ}\cdot\overrightarrow{AB}=0$

$(\mu - 1) + (10 + \mu) + \mu = 0$

$3\mu = -9 \Rightarrow \mu = -3$

$\therefore$ distance = $\sqrt{16+49+9}=\sqrt{74}$

13. A plane P contains the line of intersection of the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})$ = 6 and $\vec{r}\cdot(2\hat{i}+3\hat{j}+4\hat{k})$ = –5. If P passes through the point (0, 2, –2), then the square of distance of the point (12, 12, 18) from the plane P is

(1) 1240 (2) 620

(3) 310 (4) 155

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Equation of plane P is

$(x+y+z-6)+\lambda(2x+3y+4z+5)=0$

Plane passes through the point (0, 2, –2)

$\therefore (2-2-6)+\lambda(6-8+5)=0$

$-6+\lambda(3)=0$

$\lambda=2$

Equation of plane p is

$(x+y+z-6)+2(2x+3y+4z+5)=0$

$5x+7y+9z+4=0$

$d=\left|\frac{5\times12+7\times12+9\times18+4}{\sqrt{5^2+7^2+9^2}}\right|$

$d=\left|\frac{60+84+162+4}{\sqrt{25+49+81}}\right|$

$d=\frac{310}{\sqrt{155}}$

$d^2=\frac{310\times310}{155}=620$

14. Let f(x) be a function satisfying f(x) + f(π – x) = π^2, $\forall x \in R$. Then $\int_0^{\pi} f(x)\sin x\,dx$ is equal to

(1) $\frac{\pi^2}{4}$ (2) $\frac{\pi^2}{2}$

(3) $2\pi^2$ (4) π^2

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $f(x)+f(\pi-x)=\pi^2$

$I=\int_0^{\pi} f(x)\sin x\,dx$

Applying King's Rule

$I=\int_0^{\pi} f(\pi-x).\sin(\pi-x)dx$

$2I=\int_0^{\pi}[f(x)+f(\pi-x)]\sin x\ dx$

$2I=\int_0^{\pi}\pi^2\sin x\ dx$

$2I=\pi^2.\int_0^{\pi}\sin x\,dx$

$2I=\pi^2\times2$

$I=\pi^2$

15. If the coefficients of x^7 in $\left(ax^2+\frac{1}{2bx}\right)^{11}$ and x^{-7} in $\left(ax-\frac{1}{3bx^2}\right)^{11}$ are equal, then

(1) 64ab = 243 (2) 729ab = 32

(3) 243ab = 64 (4) 32ab = 729

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\left(ax^2+\dfrac{1}{2bx}\right)^{11}$

$T_{r+1}={}^{11}C_r(ax^2)^{11-r}.\left(\dfrac{1}{2bx}\right)^r$

$={}^{11}C_r a^{11-r}.\left(\dfrac{1}{2b}\right)^r.x^{22-2r-r}={}^{11}C_r\ a^{11-r}.\left(\dfrac{1}{2b}\right)^r.x^{22-3r}$

$\therefore 22-3r=7$

$3r=15$

$r=5$

Again $\left(ax-\dfrac{1}{3bx^2}\right)^{11}$

$T_{r+1}={}^{11}C_r(ax)^{11-r}\left(-\dfrac{1}{3bx^2}\right)^r$

$={}^{11}C_r a^{11-r}.\left(\dfrac{-1}{3b}\right)^r.x^{11-r-2r}$

$\therefore 11-3r=-7$

$3r=18$

$r=6$

Now, $\dfrac{{}^{11}C_5a^6}{32b^5}=\dfrac{{}^{11}C_6.a^5}{3^6.b^6}$

$729ab=32$

16. Among the statements

(S1): $(p\Rightarrow q)\vee((\sim p)\wedge q)$ is a tautology

(S2): $(q\Rightarrow p)\Rightarrow((\sim p)\wedge q)$ is a contradiction

(1) neither (S1) and (S2) is True

(2) only (S1) is True

(3) only (S2) is True

(4) both (S1) and (S2) are True

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $(p\to q)\vee((\sim p)\wedge q)$

p	q	$p\to q$	$\sim p\wedge q$	$(p\to q)\vee(\sim p)\wedge q)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	F	T

Not a tautology

p	q	$q\to p$	$(\sim p)\wedge q$	$(q\to p)\vee(\sim p)\wedge q)$
T	T	T	F	F
T	F	T	F	F
F	T	F	T	T
F	F	T	F	F

Not a contradiction

17. If the tangents at the points P and Q on the circle $x^2+y^2-2x+y=5$ meet at the point R $\left(\dfrac{9}{4},2\right)$, then the area of the triangle PQR is

(1) $\dfrac{13}{4}$ (2) $\dfrac{13}{8}$

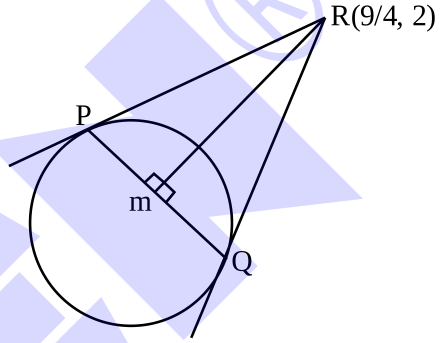
(3) $\dfrac{5}{4}$ (4) $\dfrac{5}{8}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Equation of circle is $x^2+y^2-2x+y-5=0$

$R=\dfrac{5}{2}$



Length of $PR=QR=\sqrt{S_1}$

$=\sqrt{\dfrac{81}{16}+4-\dfrac{2\times9}{4}+2-5}=\dfrac{5}{4}$

Area of triangle PQR $=\dfrac{RL^3}{R^2+L^2}=\dfrac{\dfrac{5}{2}\cdot\dfrac{125}{64}}{\dfrac{25}{4}+\dfrac{25}{16}}=\dfrac{5}{8}$

18. Let the vectors $\vec{a},\vec{b},\vec{c}$ represent three coterminous edges of a parallelopiped of volume V. Then the volume of the parallelopiped, whose coterminous edges are represented by $\vec{a},\vec{b}+\vec{c}$ and $\vec{a}+2\vec{b}+3\vec{c}$ is equal to

(1) 3V (2) 6V

(3) V (4) 2V

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $V=[\vec{a}\ \vec{b}\ \vec{c}]$

$[\vec{a},\vec{b}+\vec{c},\vec{a}+2\vec{b}+3\vec{c}]$

$=\begin{vmatrix}1&0&0\\0&1&1\\1&2&3\end{vmatrix}[\vec{a}\,\vec{b}\,\vec{c}]=1(3-2)\ V=V.$

19. If gcd (m, n) = 1 and $1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots + (2021)^2 - (2022)^2 + (2023)^2 = 1012\ m^2n$, then $m^2 - n^2$ is equal to

(1) 200 (2) 240

(3) 220 (4) 180

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $1^2-2^2+3^2-4^2+\ldots.(2021)^2-(2022)^2+(2023)^2=1012\ m^2n$

$=(1-2)(1+2)+(3-4)(3+4)+\ldots.+(2021-2022)\ (2021+2022) + (2023)^2$

$=(-1)(1+2+3+4+\ldots..+2022)+(2023)^2$

$=(-1)\cdot\frac{(2022)(2023)}{2}+(2023)^2$

$=2023(2023-1011) = 2023 \times 1012$

$m^2n=2023 = 17^2.7$

m = 17, n = 7

$m^2-n^2=17^2-7^2=240$

20. In a group of 100 persons 75 speak English and 40 speak Hindi. Each person speaks at least one of the two languages. If the number of persons, who speak only English is α and the number of persons who speak only Hindi is β, then the eccentricity of the ellipse $25(\beta^2x^2+\alpha^2y^2)=\alpha^2\beta^2$ is

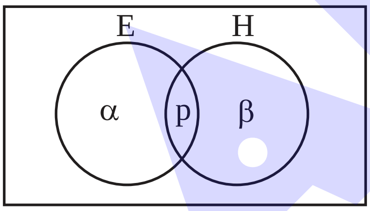
(1) $\frac{3\sqrt{15}}{12}$ (2) $\frac{\sqrt{117}}{12}$

(3) $\frac{\sqrt{119}}{12}$ (4) $\frac{\sqrt{129}}{12}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.



$\alpha+p=75$ ....(1)

$\beta+p=40$ ....(2)

$\alpha+\beta+p=100$ ....(3)

From (1), (2) and (3)

P = 15, $\alpha=60$ and $\beta=25$

Now equation of ellipse: $25\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}\right)=1$

$\frac{x^2}{144}+\frac{y^2}{25}=1$

$\Rightarrow e=\frac{\sqrt{119}}{12}$

SECTION-B

21. Let $f(x)=\frac{x}{\left(1+x^n\right)^{\frac{1}{n}}}$, $x \in R - \{-1\}$, $n \in N$, $n > 2$.

If $f^n(x)$ = (fofof ...... upto n times) (x), then $\lim\limits_{n\to\infty}\int_0^1 x^{n-2}\left(f^n(x)\right)dx$ is equal to__________

Official Ans. by NTA (0)

Allen Ans. (0)

Sol. Let $f(x)=\frac{x}{\left(1+x^n\right)^{1/x}}$, $x\in R-\{-1\}$, $n\in N$, $n>2$

$F^n(x)$ = (fofof… upto n times)(x),

then $\lim\limits_{n\to\infty}\int_0^1 x^{n-2}\left(f^n(x)\right)dx$

$f(f(x))=\frac{x}{\left(1+2x^n\right)^{1/n}}$

$f(f(f(x)))=\frac{x}{\left(1+3x^n\right)^{1/n}}$

Similarly $f^n(x)=\frac{x}{\left(1+n\cdot x^n\right)^{1/n}}$

Now $\lim\limits_{n\to\infty}\int\frac{x^{n-2}\cdot xdx}{\left(1+n\cdot x^n\right)^{1/n}}=\lim\limits_{n\to\infty}\int\frac{x^{n-1}\cdot dx}{\left(1+n\cdot x^n\right)^{1/n}}$

Now $1 + nx^n = t$

$n^2\cdot x^{n-1}dx=dt$

$x^{n-1}dx=\frac{dt}{n^2}$

$\Rightarrow \lim\limits_{n\to\infty}\frac{1}{n^2}\int_1^{1+n}\frac{dt}{t^{1/n}}$

$\Rightarrow \lim\limits_{n\to\infty}\frac{1}{n^2}\left[\frac{t^{1-\frac{1}{n}}}{1-\frac{1}{n}}\right]_1^{1+n}$

$\Rightarrow \lim\limits_{n\to\infty}\frac{1}{n(n-1)}\left((1+n)^{\frac{n-1}{n}}-1\right)$ Now let $n=\frac{1}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\left(1+\frac{1}{h}\right)^{1-h}-1}{\frac{1}{h}\frac{(1-h)}{h}}$

Using series expansion.

$\Rightarrow 0$

22. The value of tan 9° – tan 27° – tan 63° + tan 81° is _________.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. The value of tan9° –tan27°–tan63° + tan81°

$\Rightarrow$ tan9°+cot9°–tan27°–cot27°

$$\Rightarrow \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ}$$

$$\Rightarrow \frac{2\times 4}{\sqrt{5}-1} - \frac{2\times 4}{\sqrt{5}+1}$$

$$\Rightarrow 4$$

23. If the lines $\frac{x-1}{2} = \frac{2-y}{-3} = \frac{z-3}{\alpha}$

and $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{\beta}$ intersect,

then the magnitude of the minimum value of $8\alpha\beta$ is _________.

Official Ans. by NTA (18)

Allen Ans. (18)

Sol. If the lines $\frac{x-1}{2} = \frac{2-y}{-3} = \frac{z-3}{\alpha}$

And $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{\beta}$ intersect

Point on first line (1, 2, 3) and point on second line (4, 1, 0).

Vector joining both points is $-3\hat{i} + \hat{j} + 3\hat{k}$

Now vector along first line is $2\hat{i} + 3\hat{j} + \alpha\hat{k}$

Also vector along second line is $5\hat{i} + 2\hat{j} + \beta\hat{k}$

Now these three vectors must be coplanar

$$\Rightarrow \begin{vmatrix} 2 & 3 & \alpha \\ 5 & 2 & \beta \\ -3 & 1 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 2(6-\beta) - 3(15+3\beta) + \alpha(11) = 0$$

$$\Rightarrow \alpha - \beta = 3$$

Now $\alpha = 3 + \beta$

Given expression $8(3+\beta)\cdot\beta = 8(\beta^2 + 3\beta)$

$$= 8\left(\beta^2 + 3\beta + \frac{9}{4} - \frac{9}{4}\right) = 8\left(\beta + \frac{3}{2}\right)^2 - 18$$

So magnitude of minimum value = 18

24. If $(20)^{19} + 2(21)(20)^{18} + 3(21)^2(20)^{17} + \ldots\ldots + 20(21)^{19} = k(20)^{19}$, then k is equal to _________.

Official Ans. by NTA (400)

Allen Ans. (400)

Sol. If $(20)^{19} + 2(21)(20)^{18} + 3(21)^2(20)^{17} + \ldots + 20(21)^{19} = k(20)^{19}$ then k is

$$20^{19}\left(1 + 2\cdot\left(\frac{21}{20}\right) + 3\left(\frac{21}{20}\right)^2 + \ldots + 20\left(\frac{21}{20}\right)^{19}\right) = k(20)^{19}$$

$$\Rightarrow k = 1 + 2\left(\frac{21}{20}\right) + 3\left(\frac{21}{20}\right)^2 + \ldots + 20\left(\frac{21}{20}\right)^{19} \quad \ldots(1)$$

$$\Rightarrow k\left(\frac{21}{20}\right) = \frac{21}{20} + 2\cdot\left(\frac{21}{20}\right)^2 + \ldots$$

$$\ldots + 19\left(\frac{21}{20}\right)^{19} + 20\cdot\left(\frac{21}{20}\right)^{20} \quad \ldots(2)$$

Subtracting equation (2) from (1)

$$\Rightarrow k\left(\frac{-1}{20}\right) = 1 + \frac{21}{20} + \left(\frac{21}{20}\right)^2 + \ldots + \left(\frac{21}{20}\right)^{19} - 20\cdot\left(\frac{21}{20}\right)^{20}$$

$$\Rightarrow k\left(\frac{-1}{20}\right) = \frac{1\left(\left(\frac{21}{20}\right)^{20} - 1\right)}{\left(\frac{21}{20} - 1\right)} - 20\cdot\left(\frac{21}{20}\right)^{20}$$

$$\Rightarrow k\left(\frac{-1}{20}\right) = 20\left(\frac{21}{20}\right)^{20} - 20 - 20\cdot\left(\frac{21}{20}\right)^{20}$$

$$\Rightarrow k\left(\frac{-1}{20}\right) = -20$$

$$\Rightarrow k = 400$$

25. The number of 4-letter words, with or without meaning, each consisting of 2 vowels and 2 consonants, which can be formed from the letters of the word UNIVERSE without repetition is_________

Official Ans. by NTA (432)

Allen Ans. (432)

Sol. UNIVERSE

Vowels: E, I, U

Consonants: N, V, R, S

$\rightarrow {}^3C_2 \times {}^4C_2 \times 4! = 3 \times 6 \times 24 = 432$

26. The number of points, where the curve $y = x^5 - 20x^3 + 50x + 2$ crosses the x-axis, is ______.

Official Ans. by NTA (5)

Allen Ans. (5)

Sol. $y = x^5 - 20x^3 + 50x + 2$

$$\frac{dy}{dx} = 5x^4 - 60x^2 + 50 = 5(x^4 - 12x^2 + 10)$$

$$\frac{dy}{dx} = 0 \Rightarrow x^4 - 12x^2 + 10 = 0$$

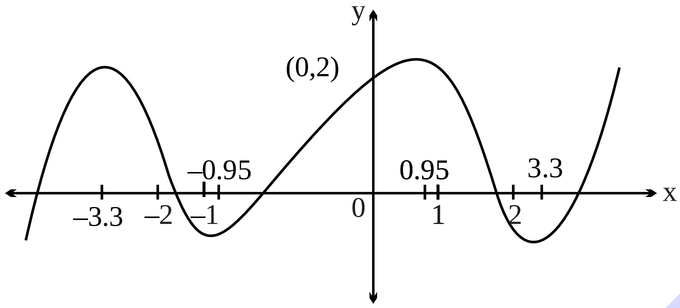
$$\Rightarrow x^2 = \frac{12 \pm \sqrt{144-40}}{2}$$

$$\Rightarrow x^2 = 6 \pm \sqrt{26} \Rightarrow x^2 \approx 6 \pm 5.1$$

$$\Rightarrow x^2 \approx 11.1, 0.9$$

$$\Rightarrow x \approx \pm 3.3, \pm 0.95$$

f(0) = 2, f (1) = + ve, f (2) = – ve
f(–1) = – ve, f (–2) = +ve



27. For $\alpha, \beta, z \in C$ and $\lambda > 1$, if $\sqrt{\lambda - 1}$ is the radius of the circle $|z - \alpha|^2 + |z - \beta|^2 = 2\lambda$, then $|\alpha - \beta|$ is equal to ______________.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. For circle :

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

$$r = \frac{|z_1 - z_2|}{2} = \frac{|\alpha - \beta|}{2} = \sqrt{\lambda - 1}$$

$$2\lambda = |\alpha - \beta|^2$$

$$|\alpha - \beta| = 2\sqrt{\lambda - 1}$$

$$|\alpha - \beta|^2 = 4\lambda - 4 = 2\lambda$$

$\lambda = 2$

$$\Rightarrow |\alpha - \beta|^2 = 4$$

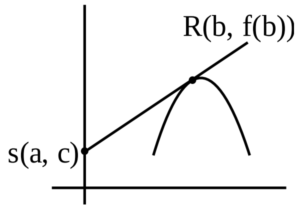
$$|\alpha - \beta| = 2$$

28. Let a curve y = f(x), $x \in (0, \infty)$ pass through the points $P\left(1, \frac{3}{2}\right)$ and $Q\left(a, \frac{1}{2}\right)$. If the tangent at any point $R(b, f(b))$ to the given curve cuts the y-axis at the point S (0, c) such that bc = 3, then $(PQ)^2$ is equal to __________________.

Official Ans. by NTA (5)

Allen Ans. (5)

Sol.



Equation of tangent at R(b, f(2)) is

$$y - f(b) = f'(b).(x - b)$$

which passes through (0, c)

$$\Rightarrow c - f(b) = f'(b).(-b)$$

$$\Rightarrow \frac{3}{b} - f(b) = f'(b).(-b)$$

$$\Rightarrow bf'(b) - f(b) = -\frac{3}{b}$$

$$\Rightarrow \frac{bf'(b) - f(b)}{b^2} = -\frac{3}{b^3}$$

$$\Rightarrow d\left(\frac{f(b)}{b}\right) = -\frac{3}{b^3} \quad \Rightarrow \frac{f(b)}{b} = \frac{3}{2b^2} + \lambda$$

Which passes through (1, 3/2)

$$\Rightarrow \frac{3}{2} = \frac{3}{2} + \lambda \Rightarrow \lambda = 0$$

$$\Rightarrow f(b) = \frac{3}{2b}$$

$$f(a) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{3}{2b} \Rightarrow b = 3$$

$$\Rightarrow c = 1 \Rightarrow Q(3, 1/2)$$

$$\Rightarrow PQ^2 = 2^2 + (1)^2 = 5$$

29. Let the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is reciprocal to that of the hyperbola $2x^2 - 2y^2 = 1$. If the ellipse intersects the hyperbola at right angles, then square of length of the latus-rectum of the ellipse is ______________.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $e_H = \sqrt{2}$

$e_E = \frac{1}{\sqrt{2}}$

Since the curves intersect each other orthogonally

The ellipse and the hyperbola are confocal

$H: \frac{x^2}{1/2} - \frac{y^2}{1/2} = 1$

$\Rightarrow$ foci $= (1, 0)$

For ellipse $a.e_E = 1$

$\Rightarrow a = \sqrt{2}$

$(e_E)^2 = \frac{1}{2} \Rightarrow 1 - \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow \frac{b^2}{a^2} = \frac{1}{2}$

$\Rightarrow b^2 = 1$

Length of L.R. $= \frac{2b^2}{a} = \frac{2}{\sqrt{2}} = \sqrt{2}$

30. If the mean and variance of the frequency distribution

x_i	2	4	6	8	10	12	14	16
f_i	4	4	α	15	8	β	4	5

are 9 and 15.08 respectively, then the value of $\alpha^2 + \beta^2 - \alpha\beta$ is ______________.

Official Ans. by NTA (25)

Allen Ans. (25)

Sol.

x_i	f_i	f_ix_i	$f_ix_i^2$
2	4	8	16
4	4	16	64
6	α	6α	36α
8	15	120	960
10	8	80	800
12	β	12β	144β
14	4	56	784
16	5	80	1280

$N = \sum f_i = 40 + \alpha + \beta$

$\sum f_i x_i = 360 + 6\alpha + 12\beta$

$\sum f_i x_i^2 = 3904 + 36\alpha + 144\beta$

$\text{Mean}(\overline{x}) = \frac{\sum f_i x_i}{\sum f_i} = 9$

$\Rightarrow 360 + 6\alpha + 12\beta = 9(40 + \alpha + \beta)$

$3\alpha = 3\beta \Rightarrow \alpha = \beta$

$\sigma^2 = \frac{\sum f_i x_i^2}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i}\right)^2$

$\Rightarrow \frac{3904 + 36\alpha + 144\beta}{40 + \alpha + \beta} - (\overline{x})^2 = 15.08$

$\Rightarrow \frac{3904 + 180\alpha}{40 + 2\alpha} - (9)^2 = 15.08$

$\Rightarrow \alpha = 5$

Now, $\alpha^2 + \beta^2 - \alpha\beta = \alpha^2 = 25$