INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS (IOQM) - 2023-24

Date: 03/09/2023

Max. Marks: 100 **SOLUTIONS** **Time allowed: 3 hours**

1. Let n be a positive integer such that $1 \le n \le 1000$. Let M_n be the number of integers in the set $X_n = \left\{\sqrt{4n+1}, \sqrt{4n+2}, ..., \sqrt{4n+1000}\right\}$. Let

$a = \max\{M_n : 1 \le n \le 1000\}$, and $b = \min\{M_n : 1 \le n \le 1000\}$.

Find $a - b$.

Ans. **(22)**

Sol. $\{\sqrt{4n+1}, \sqrt{4n+2}, ... \sqrt{4n+1000}\}$

$n = 1$

$x_n = [\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9}, \sqrt{10}, ... \sqrt{1004}]$

$n = 2$

$x_2 = [\sqrt{9}, \sqrt{10}, \sqrt{11}, ... \sqrt{1008}]$

$x_3 = [\sqrt{13}, \sqrt{14}, \sqrt{15}, ... \sqrt{1012}]$

$x_{1000} = \{\sqrt{4001}, \sqrt{4002}, \sqrt{4003}, ----- \sqrt{5000}\}$

M_n is max. when n = 1 & n = 2

$a = \max(M_n) = 29$.

$\text{Min}(M_n)$ is when $n = 1000$

$b = \text{Min}(M_n) = 7$.

$a - b = 29 - 7 = 22$

2. Find the number of elements in the set

$\{(a, b) \in N : 2 \le a, b \le 2023, \log_a(b) + 6\log_b(a) = 5\}$

Ans. **(54)**

Sol. $t + \frac{6}{t} = 5$

$\therefore$ $t^2 + 6 = 5t$

$t^2 - 5t + 6 = 0$

$t = 2, 3.$

$\log_a b = 2, \log_a b = 3$

$\therefore$ $b = a^2, b = a^3.$

(2, 4) (2, 8)

(3, 9) .

(4, 16) .

.

(12, 1728)

(32, 1024)

.

.

(33, 1089)

(44, 1936)

Possible vlaue of a = 43

Possible values of b = 11

$\therefore$ $43 + 11 \rightarrow 54$

Number of elements = 54

3. Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$$

Find the smallest possible value of β.

Ans. (23)

Sol. $\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$

$16\beta < 37\alpha$ $\qquad$ $16\alpha < 7\beta$

$\beta < \frac{37\alpha}{16}$ $\qquad$ $\beta > \frac{16\alpha}{7}$

$\frac{16\alpha}{7} < \beta < \frac{37\alpha}{16}$

For $\alpha = 1, 2, 3 \ldots\ldots 9$, $\beta \notin I^+$

At $\alpha = 10$

$22.8571 < \beta < 23.125$

$$\boxed{\beta = 23}$$

4. Let x, y be positive integers such that

$$x^4 = (x - 1)(y^3 - 23) - 1$$

Find the maximum possible value of x + y.

Ans. (07)

Sol. $x^4 = (x - 1)(y^3 - 23) - 1$

$\Rightarrow \frac{x^4 + 1}{x - 1} = y^3 - 23$

$\Rightarrow \frac{x^4 + 1}{x - 1} + 23 = y^3 \quad \Rightarrow \quad \frac{x^4 + 1 + 23x - 23}{x - 1} = y^3$

$\Rightarrow \frac{x^4 + 23x - 22}{x - 1} = y^3$

Since x and y are integers

$\therefore$ $x - 1$ will completely divide $x^4 + 23x - 22$

$$\begin{array}{c|ccccc} 1 & 1 & 0 & 0 & 23 & -22 \\ & & 1 & 1 & 1 & 24 \\ \hline & 1 & 1 & 1 & 24 & 2 \end{array}$$

remainder

There shouldn't be any remainder thus

2 must be divisible by $x - 1$ and since $x \in I$, only possible values are 2 and 3.

when x = 2

$y^3 = \frac{2^4 + 23 \times 2 - 22}{1} = \frac{46 - 6}{1} = 40$

which means $y \notin I$ when x = 3

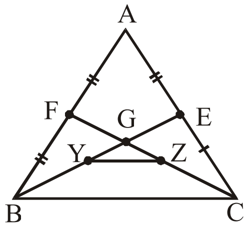
Now, $\quad y^3 = \frac{81 + 69 - 22}{2} = \frac{59 + 69}{2} = \frac{128}{2} = 64$

thus y = 4 $\qquad \therefore\ x + y = 3 + 4 = 7$

5. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle GYZ.

Ans. (10)

Sol.



Let BE = 3K, CF = 3M

$BY = \frac{3}{2}K$, $CZ = \frac{3}{2}M$

$YG = \frac{1}{2}K$, $ZG = \frac{1}{2}M$

GE = K, FG = M

$ar(\Delta ABC) = 480$

$ar(\Delta BEC) = \frac{1}{2}\, ar(\Delta ABC) = \frac{1}{2} \times 480 = 240$

$ar(\Delta BGC) = \frac{2}{3}\, ar(\Delta BEC) = \frac{2}{3} \times 240 = 2 \times 30 = 160$

In ΔBGC

$$\frac{GY}{YB} = \frac{GZ}{ZC} = \frac{1}{3}$$

$$\frac{ar(\Delta GYZ)}{ar(\Delta BGC)} = \left(\frac{1}{4}\right)^2$$

$$\frac{ar(\Delta GYZ)}{160} = \frac{1}{16}$$

$$ar(\Delta GYZ) = \frac{160}{16} = 10$$

6. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.

Ans. (16)

Sol. $\frac{(x-2)180°}{x} = n$

$180 - \frac{360}{x} = n$

As $360° = 2^3 \times 3^2 \times 5$

Total factors $= 4 \times 3 \times 2 = 24$

But for $x = 1, 2, 8, 2^3 \times 3, 2^3 \times 3^2, 2^3 \times 3^2 \times 5, 2^3 \times 3 \times 5$ won't work.

Therefore n = 24 – 8 = 16 Ans.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on oppossite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

Ans. **(48)**

Sol. Arrangement of 3, 4, 5, 6 can be done in 3! ways = 6 ways (using circular permutation)

Colouring can be done in $2 \times 2 \times 2 = 8$ ways

$\Rightarrow$ Total design are $8 \times 6 = 48$ ways.

8. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.

Ans. **(59)**

Sol. **Case I**

If we use only dominoes

For $2 \times n$ rectangle we get

recursion formula as $F(n) = F(n-1) + F(n-2)$

where $F(1) = 1, F(2) = 2, F(3) = 3, F(4) = 5, F(5) = 8, F(6) = 13, F(7) = 21$

Case II

When 2×2 tile is used

$2 \times (F(5) + F(1) \times F(4) + F(2) \times F(3))$

$2 \times (8 + 1 \times 5 + 2 \times 3) = 38$

$\Rightarrow$ Total = 21 + 38 = 59

9. Find the nubmer of triples (a, b, c) of positive integers such that,

(a) ab is a prime ;

(b) bc is a product of two primes ;

(c) abc is not divisible by square of any prime and

(d) $abc \le 30$

Ans. **(17)**

Sol. $abc \le 30$

and abc is not divisible by 4, 9, 25

So abc can take values :

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29 and 30

$\because$ ab is prime.

Case 1 : When a = 1 and b is prime number.

a = 1, b = 2, c = 3, 5, 7, 11, 13

b = 3, c = 2, 5, 7

b = 5, c = 2, 3

b = 7, c = 2, 3

b = 11 c = 2

b = 13 c = 2

there are total 14 triples of (a, b, c)

Case 2 : When b = 1 and a is prime number.

b = 1, a = 2 c = 15

a = 3 c = 10

a = 5 c = 6

there are 3 triples of (a, b, c).

from case I and case II total 17 triples of (a, b, c) are possible.

10. The sequence $\langle a_n \rangle_{n\geq 0}$ is defined by $a_0 = 1$, $a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

Ans. **(51)**

Sol. $a_0 = 1 \quad a_1 = -4 \qquad a_{n+2} = -4\,a_{n+1} - 7a_n$

$x^2 + 4x + 7 = 0$

Let x_1 and x_2 are roots

$x_1 = -2 + \sqrt{3}i \qquad x_2 = -2 - \sqrt{3}i \qquad x_1 + x_2 = -4$

$x_1 x_2 = 7$

Let $a_n = P(x_1)^n + q(x_2)^n$

at $n = 0$ $\quad a_0 = p + q = 1$

$\boxed{p + q = 1}$ ... (1)

at $n = 1$ $\quad a_1 = p(x_1) + q(x_2)$

$-4 = -2(p + q) + \sqrt{3}i(p - q)$

$\boxed{p - q = \frac{2i}{\sqrt{3}}}$ ... (2)

From equation (1) and (2) $\qquad p = \frac{1}{2} + \frac{i}{\sqrt{3}},\ q = \frac{1}{2} - \frac{i}{\sqrt{3}}$

$$a_{50}^2 - a_{49}.a_{51} = \left(p(x_1)^{50} + q(x_2)^{50}\right)^2 - \left(p(x_1)^{49} + q(x_2)^{49}\right)\left(p(x_1)^{51} + q(x_2)^{51}\right)$$

$$= 2pq(x_1x_2)^{50} - pq(x_1^{49}.x_2^{51}) - pq\ x_1^{51}.x_2^{49}$$

$$= 2 \times \frac{7}{12} \times 7^{50} - \frac{7}{12}(x_1x_2)^{49}\left(x_2^2 + x_1^2\right)$$

$$= \frac{7^{51}}{6} - \frac{7}{12}(7)^{49} \times (2) = 7^{50}$$

No. of positive integer divisors = 51

11. A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

Ans. **(14)**

Sol. $m^2 = 4n^2 - 5n + 16$

$16m^2 = 64n^2 - 80n + 256$

$16m^2 = (8n - 5)^2 + 231$

$(4m)^2 - (8n - 5)^2 = 7 \times 11 \times 3$

$(4m + 8n - 5)(4m - 8n + 5) = 7 \times 11 \times 3$

By property and for max. of 'm'

$4m + 8m - 5 = 1$

$4m - 8m + 5 = 231$

$8m \quad = 232$

$m = 29$

So n = not integer $8n = -110$

So $\quad 4m - 8m - 5 = 77$

$4m + 8m + 5 = 3$

$8m \quad = 80 \quad \Rightarrow \quad m = 10$

$n = -4$

$\therefore \quad |m - n| = |10 - (-4)| = 14$

12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_1 be the value of P(x) at x = i. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

Ans. (18)

Sol. $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$

Only possible if

$p_1 + p_2 + p_3 = 0$ or $p_1 = p_2 = p_3$

$\Rightarrow$ $36 + 14a + 6b + 3c = 0$ or $a + b + c + 1 = 8 + 4a + 2b + c = 27 + 9a + 3b + c$

$\downarrow$ Not possible to calculate

$\downarrow$ Now $3a + b = -7$ & $5a + b = -19$

$2a = -12$

$a = -6$

$b = 11$

Now $p_2 + 2p_1 - 3p_0$

$\Rightarrow$ $6a + 4b + 10$

$\Rightarrow$ $6 \times (-6) + 4 \times 11 + 10$

$= -36 + 44 = 0$

$= -36 + 54$

$= 18$

13. The ex-radii of n triangle are $10\frac{1}{2}$, 12 and 14. If the sides of the triangles are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$.

Ans. (58)

Sol. $a + b + c = p$,

$ab + bc + ca = q$

$abc = r$

$$r_1 = \sqrt{\frac{s(s-b)(s-c)}{s-a}} = \frac{21}{2}$$

$$r_2 = \sqrt{\frac{s(s-a)(s-c)}{s-b}} = 12$$

$$r_3 = \sqrt{\frac{s(s-a)(s-b)}{s-c}} = 14$$

Now, $p + q + r = a + b + c + ab + bc + ca + abc$

$= (a + 1)(b + 1)(c + 1) - 1$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r} = \frac{s}{\Delta} = \frac{p}{2\Delta}$$

$$\Rightarrow \frac{2}{21} + \frac{1}{12} + \frac{1}{14} = \frac{p}{2\Delta} = \frac{8+7+6}{84} = \frac{21}{84} = \frac{1}{4}$$

$$\Rightarrow r_1r_2 + r_2r_3 + r_3r_1 = \left(\frac{p}{2}\right)^2$$

$$\Rightarrow 126 + 168 + 147 = \frac{p^2}{4}$$

$\Rightarrow \quad 441 \times 4 = p^2$

$\Rightarrow \quad p = 2 \times 21 = 42$

$$\Rightarrow \quad \frac{42}{2\Delta} = \frac{1}{4} \quad \Rightarrow \quad \Delta = 84$$

$$\frac{\Delta}{s-a} = \frac{21}{2} \Rightarrow \frac{84}{21} \times 2 = s - a = 8$$

$$\frac{\Delta}{s-b} = 12 \Rightarrow \quad s - b = 7$$

$$\frac{\Delta}{s-c} = 14 \Rightarrow \quad s - c = 6$$

$s = 21$

$a = 13$

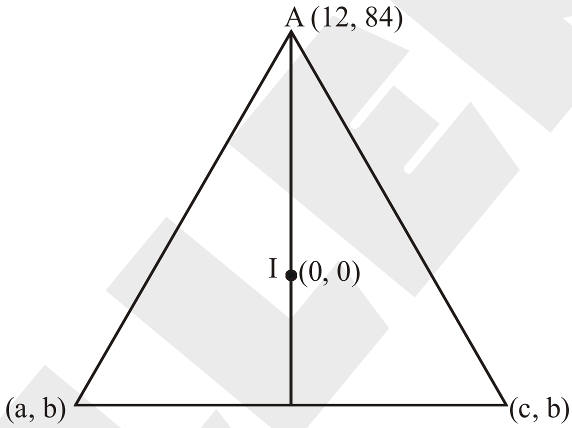
$b = 14$

$c = 15$

$\Rightarrow \quad (a + 1)(b + 1)(c + 1) = 14 \times 15 \times 16$

$\Rightarrow \quad \sqrt{(a+1)(b+1)(c+1)-1} = 57.95$

$\Rightarrow \quad 58$



14. Let ABC be a trianlge in the xy plane, where B is at the origin (0, 0). Let BC be produced to D such that BC : CD = 1 : 1. CA be produced to E such that CA : AE = 1 : 2 and AB be produced to F such that AB : BF = 1 : 3. Let G(32, 34) be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

Ans. **(40)**

Sol. Let A(a, b) C(96 – a, 72 – b) B(0, 0)

D(192 – 2a, 144 – 2b)

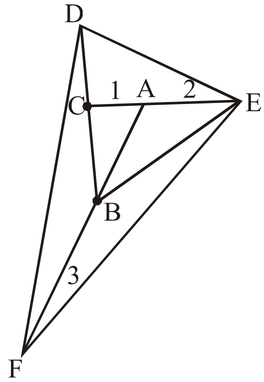
$$E\left(\frac{3a-192+2a}{3-2}, \frac{3b-144+2b}{3-2}\right) \equiv (5a - 192, 5b - 144)$$

F(–3a, –3b)

K(0, 0) G(32, 24)

$$GK = \sqrt{(32)^2 + (24)^2}$$

$$= 8\sqrt{16+9} = 40 \text{ Ans.}$$



15. Let ABCD be a unit square, Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcentre of triangle MAN, and P be the circumcentre of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of m + n.

Ans. **(03)**

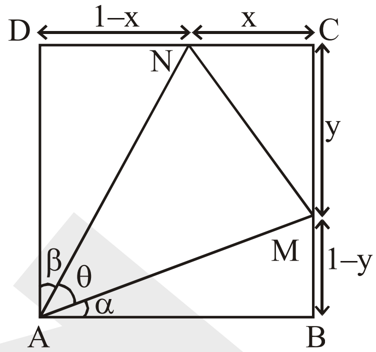
Sol. $x + y + \sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = (2 - x - y)^2$

$x^2 + y^2 = 4 + x^2 + y^2 - 4x - 4y + 2xy$

$xy = 2x + 2y - 2$

$xy - x - y = x + y - 2$



$\Rightarrow \quad \frac{2 - x - y}{x + y - xy} = 1 \qquad ......(1)$

$\tan\alpha = 1 - y$

$\tan\beta = 1 - x$

$\tan(\alpha + \beta) = \frac{1 - x + 1 - y}{1 - (1 - x)(1 - y)}$

$= \frac{2 - (x + y)}{1 - (1 - x - y + xy)}$

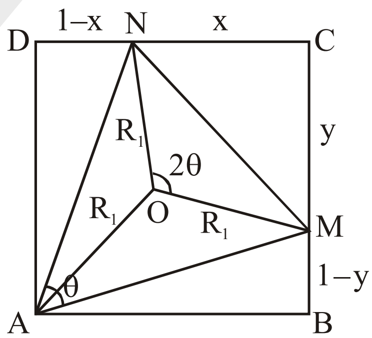
$= \frac{2 - x - y}{x + y - xy} = 1 \qquad$ [from (1)]

$\Rightarrow \alpha + \beta = 45° \qquad \Rightarrow \theta = 45°$

Let R_1 & R_2 be circumradius of ΔMCN & ΔMON.

$R_1 = OA = \frac{MN}{2\sin\theta}$

$R_2 = OP = \left(\frac{MN}{2\sin 2\theta}\right)$



$\frac{OP}{OA} = \frac{MN}{2\sin 2\theta}\frac{2\sin\theta}{MN} = \frac{1}{(2\cos\theta)} = \frac{1}{2\cos 45°}$

$\frac{OP}{OA} = \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left(\frac{OP}{OA}\right)^2 = \frac{1}{2} = \frac{m}{n}$

m + n = 3

16. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \le i < j < k \le 6$, has at least one red side, find the sum of the squares of the digits of N.

Ans. **(94)**

Sol. Number of diagonal are ${}^6C_2 - 6 = 9$

Required = $2^9 - 2^6 - 2^6 + 2^3$

= 392

sum of squares of digit is 94.

17. Consider the set

$$S = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\}$$

where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D.

Ans. **(66)**

Sol. Total selection of a, b, c, d, e are ${}^{99}C_5$

when d = 4, then total way = ${}^{3}C_3 \times {}^{95}C_1$

when d = 5, then total way = ${}^{4}C_3 \times {}^{94}C_1$

....

when d = 98, then total was = ${}^{97}C_3 \times {}^{1}C_1$

$$\Rightarrow \text{Average} = \frac{4\times {}^{3}C_3\ {}^{95}C_1 + 5\times {}^{4}C_3\ {}^{94}C_1 + \dots\dots + 98\ {}^{97}C_3\ {}^{1}C_1}{{}^{99}C_5}$$

$$\text{Numerator} = \sum_{r=3}^{97} (r+1)\,(100-(r+2))\ {}^{r}C_3$$

$$= 100 \times 4\sum_{r=3}^{97} \frac{r+1}{4}\ {}^{r}C_3 - 20\sum_{r=3}^{97} \frac{(r+1)}{4}\,\frac{(r+2)}{5}\ {}^{r}C_3$$

$$= 400\sum_{r=3}^{97} {}^{r+1}C_4 - 20\sum_{r=3}^{97} {}^{r+2}C_5$$

$$= 400 \times {}^{99}C_5 - 20 \times {}^{100}C_6$$

$$\Rightarrow \text{Average} = 400 - 20 \times \frac{{}^{100}C_6}{{}^{99}C_5}$$

$$= 400 - 20 \times \frac{100}{6}$$

$$= 400 - \frac{1000}{3} = \frac{200}{3}$$

$$= 66.6$$

= Required = 66

18. Let P be a convex polygon with 50 vertices. A set F of diagonals of P is said to be minimally friendly if any diagonal d ∈ F intersects at most one other diagonal in F at a point interior to P. Find the largest possible number of elements in a minimally friendly set F.

Ans. **(71)**

Sol. A_1A_3, $A_1\,A_4$, $A_1\,A_5$,.........., $A_1A_{49} \rightarrow 47$

A_2A_4, $A_4\,A_6$, $A_6\,A_8$,.........., $A_{48}A_{50} \rightarrow 24$

Total = 47 + 24 = 71

19. For $n \in \mathbb{N}$, let P(n) denote the product of the digits in n and S(n) denote the sum of the digits in n. Consider the set

$A = \{n \in \mathbb{N} : P(n)$ is non-zero, square free and $S(n)$ is a proper divisor of $P(n)\}$.

Find the maximum possible number of digits of the numbers in A.

Ans. **(92)**

Sol. $A = \{n \in N ; p(n) \neq 0 , \; p(n)$ is square free and $s(n)$ is proper divisor of $p(n)\}$

p(n) is square free so number n can containing digit 1, 2, 3, 5, 7 or 1, 5, 7, 6.

s(n) is proper divisor of p(n).

So max. possible value of $s(n) = 3 \times 5 \times 7 = 105$.

For making digit sum 105, n contain digit 2, 3, 5 and 7 one time and digit 1, 88 times.

$s(n) = 2 + 3 + 5 + 7 + 1 \times 88 = 105$

$p(n) = 2 \times 3 \times 5 \times 7 \times 1.........1 = 210$

max. number of digits in n = 88 + 4 = 92

20. For any finite non empty set X of integers, let max(X) denote the largest element of X and |X| denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$$\max(A) \times |B| = 12; \text{ and}$$

$$|A| \times \max(B) = 11$$

and N can be written as 100a + b where a, b are positive integers less than 100, find a + b.

Ans. **(43)**

Sol. $|A| \times \max(B) = 11 \qquad \max(A) \times |B| = 12$

Case 1 : When $|A| = 1 \qquad \max(B) = 11$

(i) $\max(A) = 12 \qquad |B| = 1$

$A = \{12\} \qquad B = \{11\} \qquad \to 1$

(ii) $\max(A) = 6 \qquad |B| = 2$

$A = \{6\} \qquad B = \{-, 11\} \qquad \to {}^{10}C_1$

(iii) $\max(A) = 4 \qquad |B| = 3$

$A = \{4\} \qquad B = \{-, -, 11\} \qquad \to {}^{10}C_2$

(iv) $\max(A) = 3 \qquad |B| = 4$

$A = \{3\} \qquad B = \{-, -, -, 11\} \qquad \to {}^{10}C_3$

(v) $\max(A) = 2 \qquad |B| = 6$

$A = \{2\} \qquad B = \{-, -, -, -, -, 11\} \to {}^{10}C_5$

(vi) $\max(A) = 1 \qquad |B| = 12$

$B = \{-, -, -, -, -, 11\} \to$ Not possible

Case 2 : When $|A| = 11 \qquad \max(B) = 1$

(i) $\max(A) = 12 \qquad |B| = 1$

$A = \{......., 12\} \qquad B = \{1\} \qquad \to {}^{11}C_{10} = 11$

(ii) $\max(A) = 6 \qquad |B| = 2 \qquad \to$ Not possible

total ordered pair $= 1 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_5 + 11$

$= 1 + 10 + 45 + 120 + 252 + 11$

$= 439$

$N = 400 + 39$

$N = 100\,a + b = 439$

$a = 4 \qquad b = 39 \qquad a + b = 4 + 39 = 43$

21. For n ∈ N, consider non-negative integer-valued functions f on {1, 2, ..., n} satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n}(i+f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

Ans. (15)

Sol. ∵ We need $\Sigma f(i)$ least we will choose n closest to 2023.

$\therefore$ for $\frac{n(n+1)}{2} + \sum f(i) = 2023$

choose n = 63

$\Rightarrow$ $2016 + \Sigma f(i) = 2023$

$\Rightarrow$ $\Sigma f(i) = 7$

Now, all we need is to partition in all possible ways.

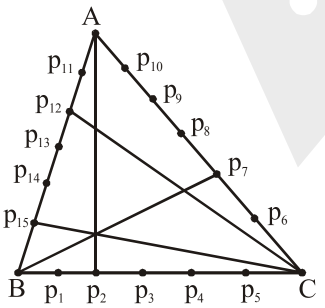
No.	No. of partition
7	1
6	1
5	2
4	3
3	4
2	3
1	1
	15

$\therefore$ Ans 15

22. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct page are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the page such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

Ans. (77)

Sol. **Case I**



As per given condition we need to divide triangle into exactly 9 region, and for flu's to happen 3 line must be concurrent as shown in the above figure.

(i.e. in a way we are choosing 3 points on three sides, such that three lines from those points are concurrent)

So basically this is ideal situation of Ova's theorem, in which product of three different ratio leads to 1.

Possible ratio on side AB, BC & CA will be of the form $\frac{m}{n}$, $\frac{n}{m}$ & 1.

i.e. $\frac{m}{n} \times \frac{n}{m} \times 1 = 1$

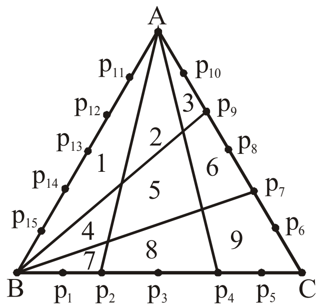
Now, we can choose the ratio 1 : 1 in 3 ways for all three sides, & other ratio can be choosen in 4 ways other two sides,

i.e. there are a $3 \times 4 + 1 = 13$ ways.

Now, fourth point can be choosen in 12C, ways

$\therefore$ Total such possibilities = $12 \times 13 = 156$ ways.

Case II



Selecting any two points on any two sides, no of ways :

$3 \times {}^5C_2 \times {}^5C_2$

$= 300$

Total cases possible

$= 300 + 156$

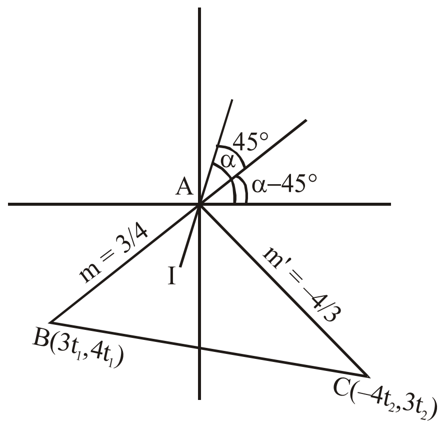
$= 456$

Sum of squares of digit = $4^2 + 5^2 + 6^2 = 16 + 25 + 36 = 77$

23. In the coordinate plane, a point is called a lattice point if both of its coordinates are integers. Let A be the point (12, 84). Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin (0, 0).

Ans. **(18)**

Sol. Use concept of shifting coordinate



Slope of $AI = \frac{84}{12} = 7$

slope of BC = $\tan(\alpha - 45°) = \frac{7-1}{1+7} = \frac{3}{4}$

radius of incircle is $\frac{AI}{\sqrt{2}} = \frac{\sqrt{12^2 + 84^2}}{\sqrt{2}} = 60$

r = 60

Use concept

$\frac{AB + AC - BC}{2} = r$

$(BC)^2 = (AC)^2 + (AB)^2$

$(BC)^2 = (AC + AB - 2r)^2$

$(5t_1 + 5t_2 - 2 \times 5 \times 12)^2 = 25(t_1^2 + t_2^2)$

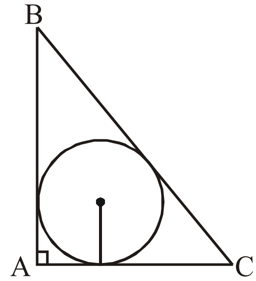
$(t_1 + t_2 - 2 \times 12)^2 = t_1^2 + t_2^2$

Put

$t_1 = x + 12$

$t_2 = y + 12$

equation become



$$\underset{\geq 0}{(x-12)}\underset{\geq 0}{(y-12)} = 2x12^2$$

Total number as triangle is equal to pair of (x, y)

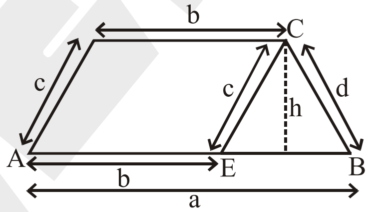
$(x - 12)(y - 12) = 2^5 \times 3^2$

No. of pair (x, y) = 6 × 3 = 18

24. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapezium whose sides are four distinct integers from the set {5, 6, 7, 8, 9, 10}.

Ans. (31)

Sol. $\frac{1}{2}h(a-b) = [BEC]$



total area : bh + [BEC]

$$b\left(\frac{2[BEC]}{a-b}\right) + [BEC]$$

$$[BEC]\left(\frac{2b+a-b}{a-b}\right)$$

$$[BEC]\left(\frac{a+b}{a-b}\right)$$

So total area is non zero if [BEC] has non zero area.

We want to find a, b, c, d ∈ {5, 6, 7, 8, 9, 10} such that [BEC] has non zero area, and are pairwise non congruent.

Note that {c, d, a – b} for a non degenerate triangle iff semiperimeter > all sides

i.e. $\frac{c+d+(a-b)}{2} > c, d, a - b$

Notice that since a, b, c, d ∈ {5, 6, 7, 8, 9, 10}

$c, d \geq 5 \geq a - b$, so we need to check

$$\frac{c+d+(a-b)}{2} > c, d$$

this becomes $|c - d| < a - b$

since exchanging c, d leads to a congruent trapezium.

Assume c > d, since they are distinct, so

$0 < c - d < a - b$ is our condition.

Now a – b can range from 1 to 5

Case I

$a - b = 1 \qquad 0 < c - d < 1 \qquad$ no solution

Case II

$a - b = 2 \qquad 0 < c - d < 2 \qquad c - d = 1$

$c = d + 1$

$(a, b) = (7, 5) \qquad d = 8$ or 9

$(a, b) = (8, 6) \qquad d = 9$

$(a, b) = (9, 7) \qquad d = 5$

$(a, b) = (10, 8) \qquad d = 5$ or $6 \qquad$ 6 solution

Case III

$a - b = 3, \qquad 0 < c - d < 3$

$c - d = 1$ or $c - d = 2$

$c = d + 1$ or $c = d + 2$

$(a, b) = (8, 5) \qquad d = 6, c = 7$

$d = 7, c = 9$

$d = 9, c = 10$

$(a, b) = (9, 6) \qquad d = 5, c = 7$

$d = 7, c = 8$

$d = 8, c = 10$

$(a, b) = (10, 7) \qquad d = 5, c = 6$

$d = 6, c = 8$

$d = 8, c = 9 \qquad$ 9 solution

Case IV

$a - b = 4 \qquad 0 < c - d < 4$

$c - d = 1$, or $c - d = 2$, or $c - d = 3$

$c = d + 1$ or $c = d + 2$, or $c = d + 3$

$(a, b) = (9, 5) \qquad d = 6, c = 7, 8$

$d = 7, c = 8, 9$

$d = 8, c = 10$

$(a, b) = (10, 6) \qquad d = 5, c = 7, 8$

$d = 7, c = 8, 9$

$d = 8, c = 9 \qquad$ 10 solution

Case V

$a - b = 5 \qquad 0 < c - d < 5$

$c - d = 1$ or $c - d = 2$ or $c - d = 3$ or $c - d = 4$

$c = d + 1$ or $c = d + 2$ or $c = d + 3$ or $c = d + 4$

$(a, b) = (10, 5) \qquad d = 6, c = 7, 8. 9$

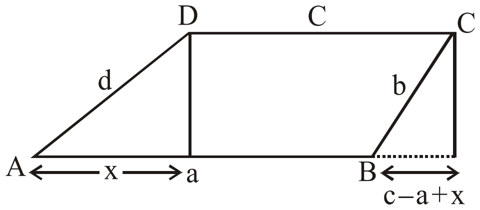
$d = 7, c = 8. 9$

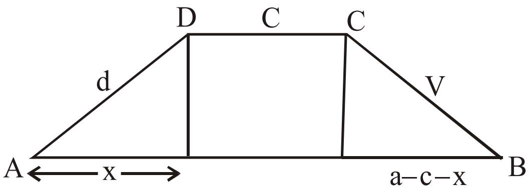
$d = 8, c = 9 \qquad$ 6 solution

Total : $0 + 6 + 9 + 10 + 6 = \boxed{31}$ trapezium.

Alternative solutions

"a > c" & "d > b" AB || CD





$d^2 - x^2 = b^2 - (c - a + x)^2$

$d^2 - x^2 = b^2 - (c - a)^2 - x^2 + 2x(a - c)$

$x = \dfrac{(a-c)^2 + d^2 - b^2}{2(a-c)}$

$d^2 - x^2 = b^2 - (a - c - x)^2$

$d^2 - x^2 = b^2 - (a - c)^2 - x^2 + 2x(a - c)$

$x = \dfrac{(a-c)^2 + d^2 - b^2}{2(a-c)}$

$\Rightarrow$ If $0 < x < d$, then a, b, c, d can form a unique trapezoid

i.e. $0 < \dfrac{(a-c)^2 + d^2 - b^2}{2c(c-a)} < d$

$\Rightarrow$ $(a - c)^2 + d^2 - 2d(a - c) - b^2 < 0$

$(a - c - d)^2 - b^2 < 0$

$(a - c - d - b)(a - c - d + b) < 0$

$\Rightarrow$ $a - c - d + b > 0$

$\Rightarrow$ $a + b > c + d$ .....(1)

& $a > c$, $d > b$

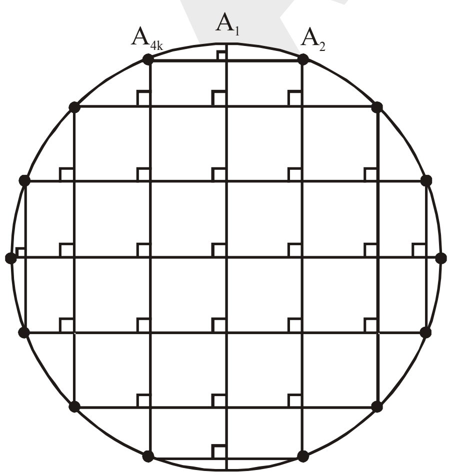
Sol. of (a, b, c, d) are

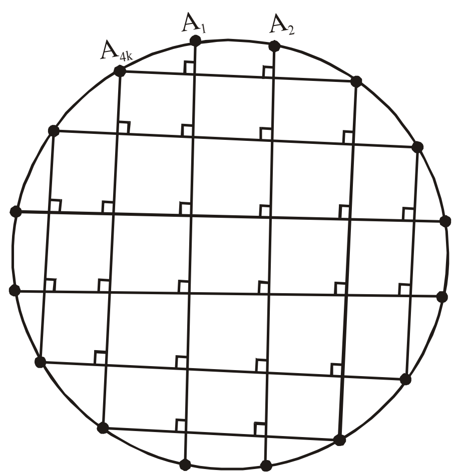
(1) 10, 8, 7, 9
(2) 10, 8, 6, 9
(3) 10, 8, 5, 9
(4) 10, 7, 6, 9
(5) 10, 7, 5, 9
(6) 10, 7, 6, 8
(7) 10, 7, 5, 8
(8) 10, 6, 5, 9
(9) 10, 6, 7, 8
(10) 10, 6, 5, 7
(11) 10, 5, 6, 8
(12) 10, 5, 6, 7
(13) 10, 5, 8, 6
(14) 10, 5, 7, 6
(15) 10, 6, 5, 8
(16) 10, 6, 8, 7
(17) 9, 8, 6, 10
(18) 9, 8, 5, 10
(19) 9, 7, 5, 10
(20) 9, 7, 5, 8
(21) 9, 7, 5, 8
(22) 9, 6, 5, 8
(23) 9, 6, 5, 7
(24) 9, 5, 6, 7
(25) 9, 5, 7, 6
(26) 8, 7, 5, 9
(27) 8, 6, 5, 7
(28) 8, 9, 5, 10
(29) 8, 9, 6, 10
(30) 7, 9, 5, 10
(31) 7, 8, 5, 9

25. Find the least positive integers n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon.

Ans. **(28)**

Sol. **Case-I**





Let n = 4k

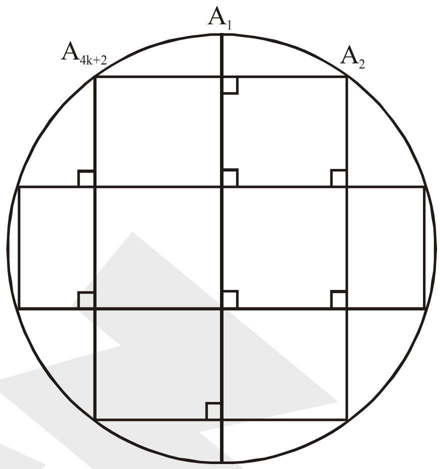
$[1 + 3 + 5 + ...+(2k-1) + ...3 + 1]k + [(2 + 4 + ... + (2K - 2)) \times 2] \times k$

$= [k^2 + (k-1)^2]\ k + 2k^2 \times (k-1) \geq 1000$

$\Rightarrow \quad k \geq 7$

$\Rightarrow \quad n \geq 28$

Case-II



n = 4k + 2

$(1 + 3 + ...+ (2k-1)) \times 2 \times (2k + 1)$

$\Rightarrow \quad 2k^2(2k + 1) \geq 1000$

$\Rightarrow \quad k^2(2k + 1) \geq 500$

$\Rightarrow \quad k \geq 7$

$\Rightarrow \quad n \geq 30$

$\Rightarrow$ Final answer is $\boxed{28}$

26. In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations 1, 2, 2^2, 2^3, ... Bens. The rules of the Government of Binary stipulate that one can not use more than two notes of any one denomination in any transaction. For example, one can given a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can given 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can given change for 100 Bens, following the rules of the Government.

Ans. (19)

Sol. No. of ways to make 100 Bens, as per Binary land government rules are as follows :

$2^6, 2^5, 2^2$
$2^6, 2^5, 2^1, 2^1$
$2^6, 2^5, 2^1, 2^0, 2^0$
$2^6, 2^4, 2^4, 2^2$
$2^6, 2^4, 2^4, 2^1, 2^1$
$2^6, 2^4, 2^4, 2^1, 2^0, 2^0$
$2^6, 2^4, 2^3, 2^3, 2^2$
$2^6, 2^4, 2^3, 2^3, 2^1, 2^1$
$2^6, 2^4, 2^3, 2^3, 2^1, 2^0, 2^0$
$2^6, 2^4, 2^3, 2^2, 2^2, 2^1, 2^1$
$2^6, 2^4, 2^3, 2^2, 2^2, 2^1, 2^0, 2^0$
$2^5, 2^5, 2^4, 2^4, 2^2$
$2^5, 2^5, 2^4, 2^4, 2^1, 2^1$
$2^5, 2^5, 2^4, 2^4, 2^1, 2^0, 2^0$
$2^5, 2^5, 2^4, 2^3, 2^3, 2^2$
$2^5, 2^5, 2^4, 2^3, 2^3, 2^1, 2^1$
$2^5, 2^5, 2^4, 2^3, 2^3, 2^1, 2^0, 2^0$
$2^5, 2^5, 2^4, 2^3, 2^2, 2^2, 2^1, 2^1$
$2^5, 2^5, 2^4, 2^3, 2^2, 2^2, 2^1, 2^0, 2^0$

Total $\boxed{19}$ possible ways.

27. A quadruple (a, b, c, d) of distinct integers is said to be balanced if a + c = b + d. Let S be any set of quadruples (a, b, c, d) where $1 \leq a < b < d < c \leq 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples is S.

Ans. (91)

Sol. At first find maximum cardinality of quadruple (a, b, c, d) ignoring the balanced one, then that is simply

$$^{20}C_4 = \frac{20\times19\times18\times17}{4} = 4845$$

but cardinality of S = {(a, b, d, c)} is given to be 4411.

So, leaving out maximum possible balanced quadruples from 4845, remaining 4845 – 4411 = 434 will lead to our answer.

i.e. question is now to find the no. of balanced quadruples in set S = {(a, b, c, d)} with $1 \le a < b < d < c \le 20$.

Now, counting balanced quadruples in S, we have

$$\begin{aligned}
a + c = b + d &= 5 \Rightarrow (1, 4), (2, 3) \rightarrow {}^2C_2\\
a + c = b + d &= 6 \Rightarrow (1, 5), (2, 4) \rightarrow {}^2C_2\\
&= 7 \Rightarrow (1, 6), (2, 5), (3, 4) \rightarrow {}^3C_2\\
&= 8 \Rightarrow (1, 7), (2, 6), (3, 5) \rightarrow {}^3C_2\\
&= 9 \Rightarrow (1, 8), (2, 7), (3, 6), (4, 5) \rightarrow {}^4C_2\\
&= 10 \Rightarrow (1, 9), (2, 8), (3, 7), (4, 6) \rightarrow {}^4C_2\\
&= 11\\
&= 12\\
&\vdots\\
&= 36 \Rightarrow (16, 20), (17, 19) \rightarrow {}^2C_2\\
&= 37 \Rightarrow (17, 20), (18, 19) \rightarrow {}^2C_2
\end{aligned}$$

$\therefore$ Total such cases are

$= 4({}^2C_2 + {}^3C_2 + ... + {}^9C_2) + {}^{10}C_2$

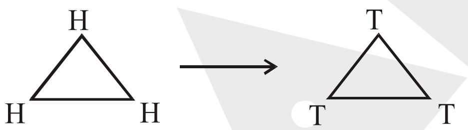
= 480 + 45 = 525 balanced quadruples.

$\therefore$ Remaining will be 525 – 434 = 91

28. On each side of an equilateral triangle with side length n units, where n is an integer, $1 \le n \le 100$, consider n – 1 points that divide the side into n equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be adjacent if the distance between them is 1 unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.

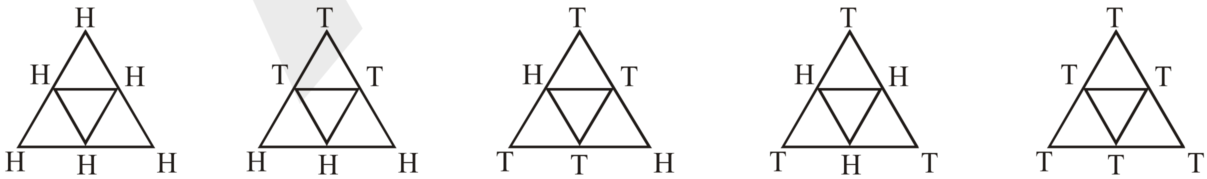
Ans. **(67)**

Sol. This can be done for $n \equiv 0, 2 \bmod 3$. Below by a triangle, we will mean three coins which are mutually adjacent. For n = 2, clearly it can be done



and

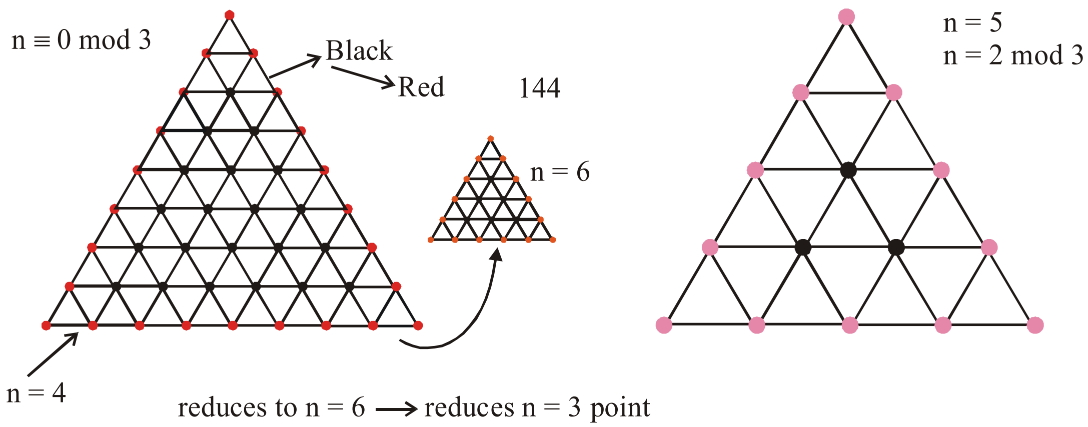
for n = 3, flip each of the four triangles.



original (I) step (II) step (III) step (IV) step

For $n \equiv 0, 2 \bmod 3$

and n > 3 flip every triangle. Then the coins at the corners are flipped once The coins on the sides (not corners) are flipped three times each. So all these coins will have tails up. The interior coins are flipped six times each and have heads up. Since the interior coins have side length n – 3, by the induction Step all of them can be flipped so to have tails up.

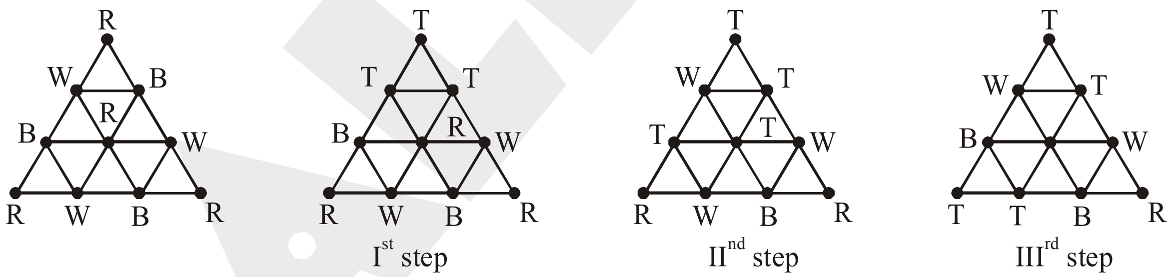


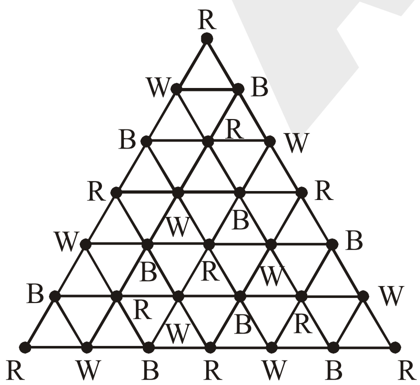
Next suppose r ≡ 1 mod 3 colour the heads of each coin red, white and blue so that adjacent coins have different colours and any three coins in a row have different colours. Then the coins in the corner have the same colour say red. [$\because$ n = 3k + 1] A simple count shows that there are one more red coins than white or blue coins, so the (odd or even) parities of the red and white coins are different in the beginning. As we flip the triangles at each turn either (a) both red and white coins increase by 1 or (b) both decrease by 1 or (c) one increase by 1 and the other decreases by 1. So the particles of the red and white coins stay different. In the case all coins are tails up the number of red and white coins could be zero and the parities would be the same so this cannot happens.

n ≡ 1 mod 3 case

<u>number of coin</u>

R	W	Step
4	3	original
3	2	(I) – 1
2	3	(II) ± 1
1	2	(III) – 1





So for n = 4, 7, ... 100

it is not possible

4 + (n – 1) 3 = 100

(n – 1) 3 = 96

(n – 1) = 32

So it is possible for r = 33

n = 1, 2, 3, 5, 6, 8, 9, ..., 98, 99

for n = 67 possible values it can be done

29. A positive integer n > 1 is called beautiful if n can be written in one and only one way as $n = a_1 + a_2 + ... + a_k = a_1 \cdot a_2 .... a_k$ for some positive integers $a_1, a_2, ...$ ak, where k > 1 and $a_1 > a_2 > .... > a_k$ (For example 6 is beautiful since 6 = 3 · 2 · 1 = 3 + 2 + 1, and this is unique. But S is not beautiful since 8 = 4 + 2 + 1 + 1 = 4 · 2 · 1 · 1 as well as 8 = 2 + 2 + 2 + 1 + 1 + = 2 · 2 · 2 · 1 · 1, so uniqueness is lost.) Find the largest beautiful number less than 100.

Ans. **(95)**

Sol. We can see that for x to be a beautiful number It must be a product of 2 primes because such a number has only two ways to express itself

largest such number is equal to 95

$$\Rightarrow \quad 95 = 19+5+\underbrace{1+1+.......+1}_{71\text{ times}}=19\times 5\times\underbrace{1\times 1\times.......\times 1}_{71\text{ times}}$$

30. Let d(m) denote the number of positive integer divisors of a positive integer m. If r is the number of integers n ≤ 2023 for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.

Ans. **(18)**

Sol. For a number to have odd divisors it must be a perfect square

n ≤ 2023. Nearest square is 44^2

But as this is an even square

$$\sum_{i=1}^{n=44^2} d(i) \rightarrow \text{even}$$

Adding odd number even times makes it even

$$\therefore \quad \sum_{i=1}^{43^2} d(i) \text{ is odd}$$

it will remain true for the numbers between

$(44^2 - 43^2) + (42^2 - 41^2) + ... + (9^2 - 1^2)$

44 + 43 + 42 + 41 +.............+ 9 + 1

$$\frac{44}{2}\times 45 = 990$$

sum of its digits = 18.