Date: 07/09/2025

INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS (SET-M1) (IOQM) - 2025-26

Max. Marks: 100 **Time allowed: 3 hours**

SOLUTIONS

1. If 60% of a number x is 40, then what is x% of 60?

Ans. (40)

Sol. 60% of x = 40

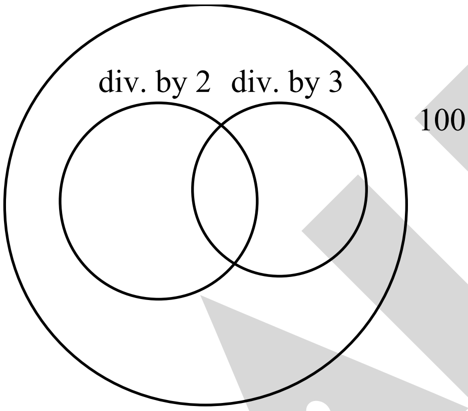
$\Rightarrow \quad \frac{60x}{100} = 40$

x% of 60 = $\frac{x}{100} \times 60 = \frac{60x}{100} = 40$

2. Find the number of positive integers n less than or equal to 100, which are divisible by 3 but are not divisible by 2.

Ans. (17)

Sol. $\left[\frac{100}{3}\right] - \left[\frac{100}{6}\right] = 33 - 16 = 17$



3. The area of an integer-sided rectangle is 20. What is the minimum possible value of its perimeter?

Ans. (18)

Sol. Area = ab = 20 a, b $\in z^+$

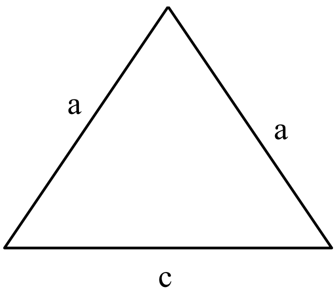
Perimeter = 2(a + b)

(a, b) = (4, 5) give rectangle with least perimeter

4. How many isosceles integer-sided triangles are there with perimeter 23?

Ans. (6)

Sol.



2a + c = 23

2a > c

$\Rightarrow$ (a, c) = (11, 1) (10, 3), (9, 5), (8, 7), (7, 9), (6, 11)

5. How many 3-digit numbers $\overline{abc}$ in base 10 are there with $a \neq 0$ and $c = a + b$?

Ans. (45)

Sol. $\overline{abc}$, a, b, c $\in$ Z

$1 \le a \le 9$

$0 \le b, c \le 9$

$c = a + b$

C	# (a, b)	{(a, b)|a + b = c}
0	0	ϕ
1	1	{(1, 0)}
2	2	{(1, 1), (2, 0)}
3	3	{(1, 2), (2, 1), (3, 0)}
4	4	{(1, 3), (2, 2), (3, 1), (4, 0)}
5	5	.
6	6	.
7	7	.
8	8	.
9	9	{(1, 8), ...(9, 0)}

$\text{Sum} = \frac{9 \times 10}{2} = 45$

6. The age of a person (in years) in 2025 is a perfect square. His age (in years) was also a perfect square in 2012. His age (in years) will be a perfect cube m years after 2025. Determine the smallest value of m.

Ans. (15)

Sol. Year of birth = y

$2025 = y + a^2$ …..(1)

$2012 = y + b^2$ …..(2)

$n^3 = a^2 + m$ …..(3)

from (1), (2) $a^2 - b^2 = 13$

$\Rightarrow$ $(a + b)(a - b) = 13 \times 1$ (as $a, b \in Z^+$)

$a = 7, b = 6$

$\Rightarrow$ $n^3 = 49 + m$

perfect able just above 49 is m = 64

$m = 64 - 49 = 15$

7. The sum of two real numbers is a positive integer n and the sum of their squares is n + 1012. Find the maximum possible value of n.

Ans. (46)

Sol. $a + b = n$

$a^2 + b^2 = n + 1012$

$a^2 - a + b^2 - b = 1012$

$$\left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 = 1012 + \frac{1}{2} = \frac{2025}{2} = \frac{(45)^2}{2}$$

let $a - \frac{1}{2} = \frac{45}{\sqrt{2}} \cos\theta$

$b - \frac{1}{2} = \frac{45}{\sqrt{2}} \cos\theta$

$n = a + b = \frac{45}{\sqrt{2}}(\sin\theta + \cos\theta) + 1$

$\max(a + b) = 45\,\frac{\sqrt{2}}{\sqrt{2}} + 1 = 46$

n = 46 is maximal integer value

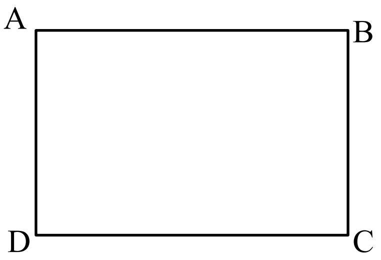
a = 23   b = 23

Ans = 46

8. A quadrilateral has four vertices A, B, C, D. We want to colour each vertex in one of the four colours red, blue, green or yellow, so that every side of the quadrilateral and the diagonal AC have end points of different colours. In how many ways can we do this?

Ans. (48)

Sol.



Every side and only diagonal AC must be of a different colour.

$\Rightarrow$ $A \neq B$ $A \neq C$ $A \neq D$

$B \neq C$ $C \neq D$

Case 1 $B \neq D$

all 4 vetrices are of a different colour

$\Rightarrow$ 4! ways to colour them = 24

Case 2   B = D

Ans =	${}^4C_1$	×	${}^3C_2$	×	2!	= 24
	↑		↑		↑	
	Ways to choose repeated colour		Ways to choose other 2 colours		Ways to colour A, C	

Ans = 48

9. Four sides and a diagonal of a quadrilateral are of lengths 10, 20, 28, 50, 75, not necessarily in that order. Which amongst them is the only possible length of the diagonal?

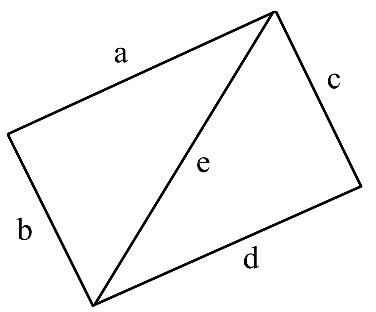
Ans. (28)

Sol. Inequalities

w.l.o.g $a < b$, $c < d$, $a < d$

Triangle $\Rightarrow$ $e < a + b$, $a - b < e$

$e < c + d$, $d - c < e$



Quadrilateral $\Rightarrow$ $d < a + b + c$

Case-1: $e = 75$

also $2e < a + b + c + d$

$\Rightarrow$ $150 < 10 + 20 + 28 + 50 = 108$ False

Case-2: $e = 50$

$\Rightarrow$ $d = 75 < a + b + c = 10 + 20 + 28 = 58$ False

Case-3: $e = 10$

There does not exist 2 pairs among 20, 28, 50, 75 such that both have difference among the nos less 10 as min $(75 - \{20, 28, 50\}) = 25 > 10$

Case-4: $e = 20$

There does not exist 2 pairs among 10, 28, 50, 75 such that both pairs have difference among the nos. less than 10 as min $(75 - \{10, 28, 50\} = 25 > 20$

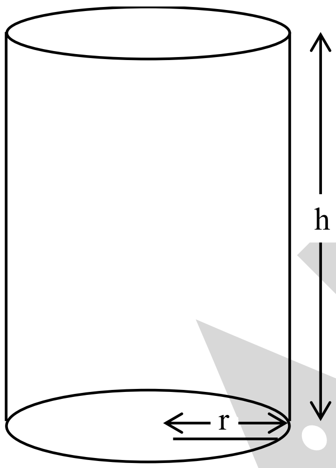
Thus $e = 28$, $d = 75$, $c = 50$, $a = 20$, $b = 10$

10. The height and the base radius of a closed right circular cylinder are positive integers and its total surface area is numerically equal to its volume. If its volume is $k\pi$ where k is a positive integer, what is the smallest possible value of k?

Ans. (54)

Sol. $\pi r^2 h = 2\pi r(r + h) = k\pi$

$\Rightarrow$ $rh = 2(r + h)$



$rh - 2r - 2h + 4 = 4$

$(r - 2)(h - 2) = 4$

$= 1 \times 4$

2×2

4×1

$(r, h) \in \{(3, 6), (4, 4), (6, 3)\}$

$K = r^2 h$

$K_{min} = 3^2 \times 6 = \boxed{54}$

11. Consider a fraction $\frac{a}{b} \neq \frac{3}{4}$, where a, b are positive integers with $\gcd(a, b) = 1$ and $b \leq 15$. If this fraction is chosen closest to $\frac{3}{4}$ amongst all such fractions, then what is the value of $a + b$?

Ans. (26)

Sol. $\frac{a}{b} \neq \frac{3}{4}$ gcd (a, b) = 1 $b \leq 15$

Consider $a = 3k + x$ $b = 4k + y$ $k, x, y \in Z$

$4k + y \leq 15$ $4x \neq 3y$

$$m = \left|\frac{a}{b} - \frac{3}{4}\right| = \left|\frac{12k + 4x - (12k + 3y)}{4b}\right| = \left|\frac{4x - 3y}{4b}\right|$$

In order to minimum m consider $b = 15$, $y = -1$, $x = -1$, $k = 4$

$\Rightarrow |4x - 3y| = 1$

$\Rightarrow a = 11$ $a + b = \boxed{26}$

Smallest possible value of $|4x - 3y|$ is 1 and $(x, y) = (-1, -1)$ is valid largest possible value of denominator is dependent only on b and $b_{max} = 15$

12. Consider five-digit positive integers of the form $\overline{abcab}$ that are divisible by the two digit number $\overline{ab}$ but not divisible by 13. What is the largest possible sum of the digits of such a number?

Ans. (33)

Sol. $13 \times \overline{abcab}$

Since $1001 = 7 \times 11 \times 13$

$\overline{abcab} = (1001) \times \overline{ab} + 100C = 100C \bmod 13$

Since $100C \neq 0 \bmod 13$

$\forall\, c \in \{1, 2, \ldots\ldots, a\}$

c can be any digit

$\overline{ab} \mid \overline{abcab} \Rightarrow \overline{abcab} = 0 \bmod \overline{ab}$

$\Rightarrow 100C = 0 \bmod \overline{ab}$

2 digit factors of 100C with highest digit sum is 75 when c = 9 largest sum of digits is $S(75975) = \boxed{33}$

13. Three sides of a quadrilateral area $a = 4\sqrt{3}, b = 9$ and $c = \sqrt{3}$. The sides a and b enclose an angle of 30°, and the sides b and c enclose an angle of 90°. If the acute angle between the diagonals is x°, what is the value of x?

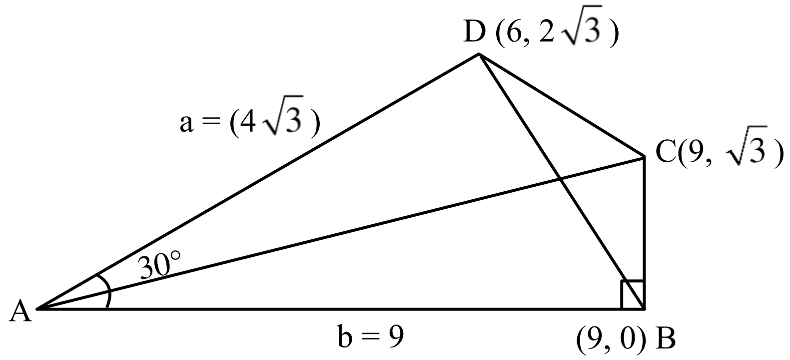
Ans. (60)

Sol. A = (0, 0)

B = (9, 0)

C = $(9, \sqrt{3})$

$D = \left(4\sqrt{3} \times \cos 30°, 4\sqrt{3} \sin 30°\right) = \left(6, 2\sqrt{3}\right)$



$$m_{AC} = \frac{\sqrt{3}}{9} = \frac{1}{3\sqrt{3}} \qquad m_{BD} = \frac{0-2\sqrt{3}}{9-6} = -\frac{2}{\sqrt{3}}$$

$$\tan x = \left|\frac{m_{AC} - m_{BD}}{1 + m_{AC} \times m_{BD}}\right| = \left|\frac{\frac{1}{3\sqrt{3}} + \frac{2}{\sqrt{3}}}{1 - \frac{1}{3\sqrt{3}} \times \frac{2}{\sqrt{3}}}\right| = \left|\frac{\frac{1+6}{3\sqrt{3}}}{\frac{9-2}{9}}\right|$$

$= \sqrt{3}$

$x = 60°$

14. A function f is defined on the set of integers such that for any two integers m and n,

$$f(mn + 1) = f(m)\, f(n) - f(n) - m + 2$$

holds and f(0) = 1. Determine the largest positive integer N such that $\sum_{k=1}^{N} f(k) < 100$.

Ans. (12)

Sol. $f(mn + 1) = f(m)\, f(n) - f(n) - m + 2, \quad f(0) = 1$

let m, n = 0

$\Rightarrow f(1) = f(0)\, f(0) - f(0) + 2 \qquad \Rightarrow f(1) = 2$

Let n = 0

$\Rightarrow f(1) = f(m)\, f(0) - f(0) - m + 2$

$\Rightarrow f(m) = m + 1$

$f(mn + 1) = mn + 2$

$f(m)\ f(n) - f(n) - m + 2 = (m + 1)(n + 1) - n - 1 - m + 2$

$= mn + m + n + 1 - n - 1 - m + 2$

$= mn + 2$

$$\sum_{k=1}^{N} f(k) = \sum_{K=1}^{N} (k+1) = \frac{(N+1)(N+2)}{2} - 1 < 100$$

$N^2 + 3N < 200$

for N = 13 $\quad N^2 + 3N = 169 + 39 = 208 > 200$

for N = 12 $\quad N^2 + 3N = 12 \times 15 = 180 < 200$

Thus N = $\boxed{12}$

15. There are six coupons numbered 1 to 6 six envelopes, also numbered 1 to 6. The first two coupons are placed together in any one envelope. Similarly, the third and the fourth are placed together in a different envelope, and the last two are placed together in yet another different envelope. How many ways can this be done if no coupon is placed in the envelope having the same number as the coupon?

Ans. (40)

Sol. This question is similar to derangements first 2 coupon can go in any 4 of the envelops.

w.l.o.g consider 1^{st} and 2^{nd} coupons goes in 3^{rd} envelope (4 ways)

Case-1 : 3^{rd} and 4^{th} coupons go in 1^{st} or 2^{nd} envelope w.l.o g they go in 1^{st} envelope (2 ways)

$\Rightarrow$ 5^{th} and 6^{th} coupon have 2 choices 2^{nd} or 4^{th} envelope

Case-2 : 3^{rd} and 4^{th} coupons go in 5^{th} or 6^{th} envelope

w.l.o.g they go in 5^{th} envelope (2 ways)

$\Rightarrow$ 5^{th} and 6^{th} coupon have 3 $\quad 1^{st}$, 2^{nd} or 4^{th} envelope

$\Rightarrow$ Ans = $4 \times (2 \times 2 + 2 \times 3) = \boxed{40}$

16. Let f(x) and g(x) be two polynomials of degree 2 such that

$$\frac{f(-2)}{g(-2)}=\frac{f(3)}{g(3)}=4.$$

If g(5) = 2, f(7) = 12, g(7) = – 6, what is the value of f(5)?

Ans. (22)

Sol. $f(x) - 4g(x) = k(x + 2)(x - 3)$

for x = 7 $12 + 24 = k(7 + 2)(7 - 3)$

$36 = k(9)(4)$

$k = 1$

$\Rightarrow$ $f(5) - 4 \times g(5) = (5 + 2)(5 - 3)$

$f(5) = 7 \times 2 + 4 \times 2 = \boxed{22}$

17. MTAI is a parallelogram of area $\frac{40}{41}$ square units such that MI = 1/MT. if d is the least possible length of the diagonal MA, and $d^2 = \frac{a}{b}$, where a, b are positive integers with gcd(a, b) = 1, find |a – b|.

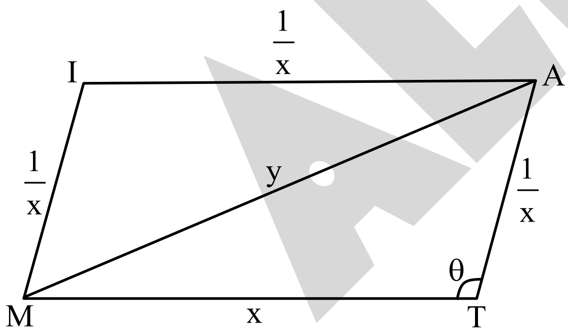
Ans. (23)

Sol. $[MTAI] = \frac{40}{41}$

d = mA

$\frac{1}{x}+x > mA$

$\boxed{MA < 2}$



$[MATI] = 2\times\frac{1}{2}\cdot x\cdot\frac{1}{x}\cdot\sin\theta$

$= \boxed{\sin\theta = \frac{40}{41}}$

$\therefore \boxed{\cos\theta = \frac{9}{41}}$

$\therefore \cos\theta = \dfrac{x^2+\frac{1}{x^2}-y^2}{2\cdot x\times\frac{1}{x}} = \frac{9}{41}$

$x^2+\frac{1}{x^2}-y^2 = \frac{9}{41}\times 2$

$y^2 = x^2 + \frac{1}{x^2} - \frac{18}{41}$

$y^2 \geq 2 - \frac{18}{41}$

Least value of $y^2 = 2 - \frac{18}{41}$

$y^2 = \frac{64}{41}$

$\therefore\ |a - b| = |64 - 41| = \boxed{23}$

18. Let N be the number of nine-digit integers that can be obtained by permuting the digits of 223334444 and which have at least one 3 to the right of the right-most occurrence of 4. What is the remainder when N is divided by 100?

Ans. (40)

Sol. Ways to arrange three 3s and four 4s

Where right most among them is 3

$\Rightarrow \quad \frac{6!}{2!\,4!} = \frac{6 \times 5}{2} = \boxed{15}$

Ways to choose the position of 2s in the 9 digit number

$= {}^9C_2 = \frac{9 \times 8}{2} = 36$

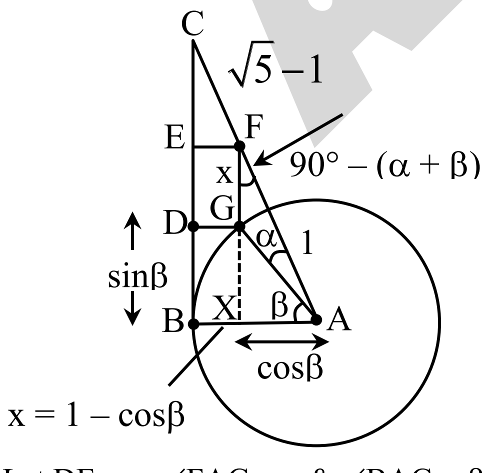
Total ways = 36 × 15 = 540

Ans = 40

19. In triangle ABC, $\angle B = 90°$, AB = 1 and BC = 2. On the side BC there are two points D and E such that E lies between C and D and DEFG is a square, where F lies on AC and G lies on the circle through B with centre A. If the area of DEFG is $\frac{m}{n}$ where m and n are positive integers with gcd(m, n) = 1, what is the value of m + n?

Ans. (29)

Sol.



Let DE = x, $\angle FAG = x$ & $\angle BAG = \beta$

and Let $FG \cap AB = X$

In ΔABC, $\tan(\alpha + \beta) = 2$

In Δ AFX

$\tan(\alpha + \beta) = \frac{FX}{AX}$

$\tan(\alpha + \beta) = \dfrac{x + \sin\beta}{\cos\beta}$

$\Rightarrow \quad 2\cos\beta = 1 - \cos\beta + \sin\beta$

$\Rightarrow \quad 3\cos\beta - 1 = \sin\beta$

$\Rightarrow \quad 9\cos^2\beta + 1 - 6\cos\beta = 1 - \cos^2\beta$

$\Rightarrow \quad 10\cos^2\beta - 6\cos\beta = 0$

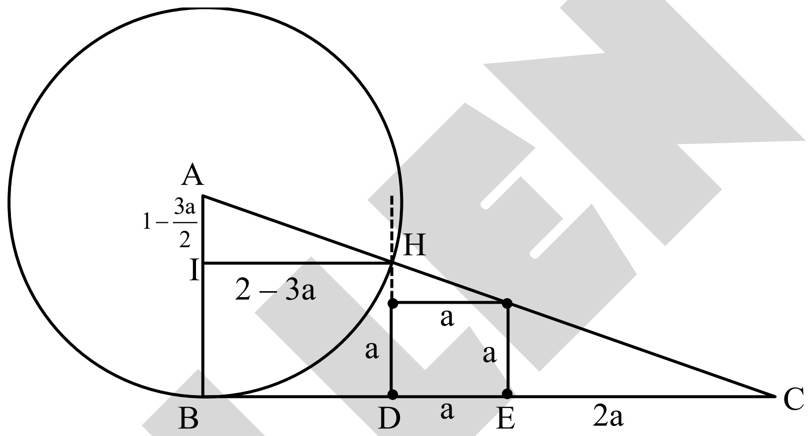
$\Rightarrow \quad \cos\beta = \dfrac{3}{5}$

$\Rightarrow \quad x = 1 - \dfrac{3}{5} = \dfrac{2}{5}$

$\Rightarrow \quad$ Area of square DEFG $= x^2 = \dfrac{4}{25} = \dfrac{m}{n}$

$m + n = 29$

OR



B = (0, 0)

A = (0, 1)

G ≡ (2 – 3a, a)

AG = 1

$\Rightarrow \quad (2-3a)^2 + (1-a)^2 = 1$

$10a^2 - 14a + 4 = 0$

$5a^2 - 7a + 2 = 0$

$a = 1$ (rejected) or $\dfrac{2}{5}$

$\therefore \quad \text{Area} = \left(\dfrac{2}{5}\right)^2 = \dfrac{4}{25}$

Ans : 4 + 25 = 29

20. Let f be the function defined by

$$f(n) = \text{remainder when } n^n \text{ is divided by } 7,$$

for all positive integers n. Find the smallest positive integer T such that f(n + T) = f(n) for all positive integers n.

Ans. (42)

Sol. $n^7 = n \bmod 7$ (FLT)

Consider r = remainder when n is divided by 7

Consider q = 1 + (remainder when n – 1 divided by 6)

$\Rightarrow$ $n^n = r^q \bmod 7$

$q \downarrow r \rightarrow$	0	1	2	3	4	5	6
1	0	1	2	3	4	5	6
2	0	1	4	2	2	4	1
3	0	1	1	6	1	6	6
4	0	1	2	4	4	2	1
5	0	1	4	5	2	3	6
6	0	1	1	1	1	1	1

For unique values of r mod 7 and q mod 6 there exists a unique n mod 42

Thus f(n + 42) = f(n)

Ans = T = $\boxed{42}$

21. For some real numbers m, n and a positive integer a, the list $(a + 1)n^2$, m^2, $a(n + 1)^2$ consists of three consecutive integers written in increasing order. What is the largest possible value of m^2?

Ans. (49)

Sol. $(a + 1)n^2$, m^2, $a(n + 1)^2$ are consecutive integer in increasing order.

$an^2 + 2an + a - an^2 - n^2 = 2$

$n^2 - 2an - a + 2 = 0$

$(n - a)^2 = a^2 + a - 2 = (a - 2)(a + 1)$

$n = a \pm \sqrt{a^2 + a - 2}$

$(a + 1)n^2 = (a + 1)\left(2a^2 + a - 2 \pm 2a\sqrt{a^2 + a - 2}\right) \in Z$

$\Rightarrow a^2 + a - 2 = k^2 \quad \exists\, k \in Z$

$\left(a + \frac{1}{2}\right)^2 - k^2 = \frac{9}{4}$

$(2a + 1)^2 - (2k)^2 = 9$

$\Rightarrow$ k = 2 and a = 2

$\Rightarrow$ n = 2 ± 2 = 4, 0

let n = 4 $\Rightarrow$ $(a + 1)n^2 = 48$

$\Rightarrow$ $a(n + 1)^2 = 50$

$\Rightarrow$ $m^2 = 49$

Let n = 0 $\Rightarrow$ $(a + 1)n^2 = 0$

$\Rightarrow$ $a(n + 1)^2 = 2$

$\Rightarrow$ $m^2 = 1$

$\Rightarrow$ k = 0 a = 1

$\Rightarrow$ n = 1

$(a + 1)n^2 = 2$

$a(n + 1)^2 = 4$

$m^2 = 3$

$m^2_{max} = \boxed{49}$

22. There are m blue marbles and n red marbles on a table. Armaan and Babita play a game by taking turns. In each turn the player has to pick a marble of the colour of his/her choice. Armaan starts first, and the player who picks the last red marble wins. For how many choices of (m, n) with $1 \le m, n \le 11$ can Armaan force a win?

Ans. (66)

Sol. Sun Starting at (m, n) we con move to (m – 1, n) or (m, n – 1)

For n = 1 : Armaan always wins, as he can just pick red – 11 cases ($1 \le m \le 11$)

For even n > 1 and odd m : Arman can force opponent to pick from even pile of red marble, after no blue ones are left.

$\rightarrow$ (n = {2, 4, 6, 8, 10}, m = {1, 3, 5, 7, 9, 11} $\rightarrow$ 30 cases

For odd n > 1 and m : even : Taking a red marble first make this an even number red marble scenario for opponent, which Armaan can force his opponent with, can guarantee a win if the number of blue marbles, m is even.

$\rightarrow$ (n = {3, 5, 7, 9, 11)}, m = {2, 4, 6, 8, 10})

$\rightarrow$ 25 cases

Ans. 66

OR

Let (b, r) be a game state with r red marbles and b blue marbles. Thus

When 1 red marble remains the next player will win

$\Rightarrow$ $(m, 1) \in NP \; m \in Z^+$

$\downarrow$

Next player win states

lets define a move $M : GS \rightarrow P(GS)$

$M((b, r)) = \{(b, r-1), (b-1, r)\}$

$x \in P$ iff $M(x) \subseteq NP$

Also $x \in NP$ iff $\exists \; y \in P$ such that $y \in m(x)$

$(0, 1) \in NP \Rightarrow (0, 2) \in P$

$\Rightarrow$ $(0, 2k-1) \in NP$ and $(0, 2K) \in P \quad \forall K \in Z^+$

$\Rightarrow$ $(1, 2K+1) \in P$ and $(1, 2K) \in NP \; \forall K \in Z^+$

$\Rightarrow$ $(2l-1, 2K+1) \in P \quad (2l-1, 2K) \in NP \; K, \forall l \in Z^+$

$(2l, 2K+1) \in NP \quad (2l, 2K) \in P$

We want the count of $(m, n) \in NP$ such that

$1 \le m, n \le 11$

If n = 1 $(m, 1) \in NP$ 11 cases

if $n \ne 1$ $(m, n) \in NP$ iff $m + n = 1 \bmod 2$

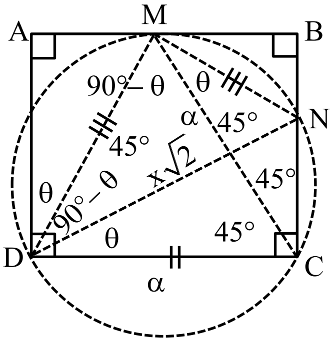
$\frac{10 \times 11}{2} = 55$ cases

Total 66 ways Armaan can win

23. Let ABCD be a rectangle and let M, N be points lying on sides AB and BC, respectively. Assume that MC = CD and MD = MN, and that points C, D, M, N lie on a circle. If $(AB/BC)^2 = m/n$ where m and n are positive integers with gcd(m, n) = 1, what is the value of m + n?

Ans. (3)

Sol.



$45° + \theta = 90° - \theta$

$2\theta = 45°$

$\theta = \dfrac{45°}{2}$

$\cos\theta = \dfrac{BC}{x} \quad \Rightarrow \quad BC = x \cos 22.5°$

$\cos\theta = \dfrac{AB}{x\sqrt{2}} \quad \Rightarrow \quad AB = x\sqrt{2} \cos 22.5°$

$\dfrac{AB}{BC} = \dfrac{x\sqrt{2}\cos 22.5°}{x\cos 22.5°} = \sqrt{2}$

$\left(\dfrac{AB}{BC}\right)^2 = \dfrac{2}{1} \quad \Rightarrow \quad 3$

24. Let $P(x) = x^{2025}$, $Q(x) = x^4 + x^3 + 2x^2 + x + 1$. Let R(x) be the polynomial remainder when the polynomial P(x) is divided by the polynomial Q(x). Find R(3).

Ans. (53)

Sol. $P(x) = x^{2025}$

$Q(x) = x^4 + x^3 + 2x^2 + x + 1 = (x^2 + x + 1)(x^2 + 1)$

$P(x) = Q(x)\, q(x) + R(x)$

$P(w) = R(w) = 1$

$P(w^2) = R(w^2) = 1$

$P(i) = R(i) = i$

$P(-i) = R(-i) = -i$

$\Rightarrow \quad R(x) = (x^2 + 1)(Ax + B) + (x^2 + x + 1)(Cx + D)$

$\Rightarrow \quad (1 + w^2)(Aw + B) = 1$

$\Rightarrow \quad -Aw^2 - Bw = 1$

$\Rightarrow \quad A = B$

$\Rightarrow \quad i(Ci + D) = i$

$\Rightarrow \quad iD - C = i$

$\Rightarrow \quad D = 1, C = 0$

$\Rightarrow \quad R(x) = (x^2 + 1)(x + 1) + (x^2 + x + 1)(1)$

$= x^2 + x^2 + x + 1 + x^2 + x + 1$

$= x^3 + 2x^2 + 2x + 2$

$R(3) = 27 + 18 + 6 + 2 = 53$

25. For how many numbers n in the set {1, 2, 3, …, 37} can we split the 2n numbers 1, 2,…., 2n into n pairs $\{a_i, b_i\}$, $1 \le i \le n$, such that $\prod_{i=1}^{n}(a_i + b_i)$ is a square?

Ans. (36)

Sol. **Case-1** : $n \to$ even

Split as following.

{1, 2n}, {2, 2n–1}, {3, 2n – 2}, ………., {n, n + 1}

$\prod_{i=1}^{n}(a_i + b_i) = (2n+1)^n$, which is a perfect square.

I = 1

Sq n = {2, 4, 6, ……., 36} $\rightarrow$ total 18 solutions.

Case-2 : n $\rightarrow$ odd.

n = 1 $\rightarrow$ only one pair {1, 2} $\rightarrow$ not a perfect sequence.

n = 3 $\rightarrow$ {1, 5}, {2, 4}, {3, 6}

given $\prod_{i=1}^{3}(a_i + b_i) = 6 \times 6 \times 9 = 324 = 18^2$

Sq n = 3 is a solution.

For n ≥ 3

We can pair up two terms to get sum = 9 and all other paired to get equal sum in two pairs.

Ex : for n = 7 : {1, 2, 3,………, 14}

{3, 6}, {1, 14}, {2, 13}, {4, 12}, {5, 11}, {7, 10}, {8, 9}

$\therefore$ solution exists for all except n = 1.

26. Consider a sequence of real numbers of finite length. Consecutive four term averages of this sequence are strictly increasing, but consecutive seven term averages are strictly decreasing. What is the maximum possible length of such a sequence?

Ans. (10)

Sol. $a_i + a_{i+1} + a_{i+2} + a_{i+3} < a_{i+1} + a_{i+2} + a_{i+3} + a_{i+4}$

$\Rightarrow a_{i+4} > a_i \ \forall i$

Similarly, $a_{i+7} < a_i \ \forall i$

For n = 11

$a_1 < a_5 < a_9$	$a_1 > a_8$
$a_2 < a_6 < a_{10}$	$a_2 > a_9$
$a_3 < a_7 < a_{11}$	$a_3 > a_{10}$
$a_4 < a_8$, $a_5 < a_9$	$a_4 > a_{11}$
$a_6 < a_{10}$, $a_7 < a_{11}$	

$\therefore a_{11} > a_7 > a_3 > a_{10} > a_6 > a_2 > a_9 > a_5 > a_1 > a_8 > a_4 > a_{11}$

Contradiction

Sq. n ≥ 11 is not possible.

for n = 10	$a_1 > a_8$
$a_1 < a_5 < a_9$, $a_2 < a_6 < a_{10}$	$a_2 > a_9$
$a_3 < a_7$, $a_4 < a_8$, $a_5 < a_9$, $a_6 < a_{10}$	$a_3 > a_{10}$

$a_7 > a_3 > a_{10} > a_6 > a_2 > a_9 > a_5 > a_1 > a_8 > a_4$

Sq. n = 10 is largest possible sequence.

27. Find the number of ordered triples (a, b, c) of positive integers such that 1 ≤ a,b,c ≤ 50 which satisfy the relation

$$\frac{\text{lcm}(a,c) + \text{lcm}(b,c)}{a+b} = \frac{26c}{27}.$$

Here, by lcm (x, y) we mean the LCM, that is, least common multiple of x and y.

Ans. (40)

Sol. $GCD(a, c) = G_1$, $GCD(b, c) = G_2$, $LCM(a, c) = L_1$, $LCM(b, c) = L_2$

So, $L_1G_1 = ac$, $L_2G_2 = bc$

and assume $a = G_1x$, $b = G_2y$ where $GCD(x, y) = 1$

$$\frac{LCM(a,c)+LCM(b,c)}{a+b}=\frac{\frac{ac}{G_1}+\frac{bc}{G_2}}{a+b}=\frac{\frac{G_1xc}{G_1}+\frac{G_2yc}{G_2}}{a+b}=\frac{2bc}{27}$$

$\Rightarrow$ $27x + 27y = 26G_1x + 26G_2y$

$x(27 - 26G_1) + y(27 - 26G_2) = 0$

If $27 - 26G_1 > 0$ then $27 - 26G_2 < 0$

$\Rightarrow$ $27 > 26G_1$ $27 < 26G_2$

So, $G_1 \geq 1$ $G_1 = 1$

$(a, c) = 1$

$a = x$, $b = G_2y$

$a = x = y\,(26G_2 - 27)$

Given $a \leq 50$

$26G_2 - 27 \leq 50$

$G_2 = \leq \frac{77}{26}$

$G_2 = 1, 2$

So if $G_1 = 1$, this means $G_2 = 2$

Now, $G_1 = 1$, $G_2 = 2$

$\Rightarrow$ $GCD(a, c) = 1$ and $GCD(b, c) = 2$

$a = 25y \leq 50$, C is even.

if $y = 1 \rightarrow$ $a = 25$, $b = 2$, $GCD(25, c) = 1$

$\Rightarrow$ Total possible values of C in this case is 20 and after reversing a and b will get 20 more cases

Now, $20 \times 2 = 40$

28. Assume a is a positive integer which is not a perfect square. Let x, y be non-negative integers such that $\sqrt{x-\sqrt{x+a}}=\sqrt{a}-y$. What is the largest possible value of a such that $a < 100$?

Ans. (91)

Sol. $\sqrt{x-\sqrt{x+a}}=\sqrt{a}-y$ $x, y \in w$ $a \in I$

By squaring both side: $a \neq$ perfect square.

$x-\sqrt{x+a}=a+y^2-2y\sqrt{a}$

Case-1 : If we take $y \neq 0$ $2y\sqrt{a}$ = Irrational and $a + y^2$ = rational

By comparison :

$x = a + y^2$ …….(1)

and $\sqrt{x+a}=2y\sqrt{a}$

$x + a = 4y^2a$ …….(2)

By solving equation (1) and (2):

$$a = \frac{y^2}{4y^2 - 2}$$

$$a = \frac{1}{4}\left[1 + \frac{2}{4y^2 - 2}\right]$$

For $y \in w$ only a = 0 we are getting which is not positive integer.

Case-2 : If $y = 0$

$$x - \sqrt{x + a} = a$$

$$x - a = \sqrt{x + a}$$

$$x^2 - 2ax + a^2 = x + a$$

$$x^2 - (2a + 1)x + a^2 - a = 0$$

$\because$ x is an integer

D = perfect square

$D = (2a + 1)^2 - 4(a^2 - a) = k^2$

$D = 8a + 1 = k^2$

a = 91 satisfy this condition

So largest possible value of a = 91

29. A regular polygon with $n \geq 5$ vertices is said to be colourful if it is possible to colour the vertices using at most 6 colours such that each vertex is coloured with exactly one colour, and such that any 5 consecutive vertices have different colours. Find the largest number n for which a regular polygon with n vertices is **not** colourful.

Ans. (19)

Sol. Let the colors be a, b, c, d, e, f. Let sequence a,b,c,d,e, be called F and the sequence a,b,c,d,e,f be called S. If $n>0$ is representable in the form $5i+ 6j$, for $i,j\geq 0$, then n satisfies the conditions of the problem: we may place i consecutive F sequences, followed by j consecutive S sequences, around the polygon. Setting j = 0, 1, 2, 3, or 4, we find that n may equal any number of the form 5i, 5i+ 6, 5i+ 12, 5i+ 18, or 5i+ 24. The only numbers greater than 4 not of this form are 7, 8, 9, 13, 14, and 19. We show that none of these numbers has the required property. Assume for a contradiction that a coloring exists for n equal to one of 7, 8, 9, 13, 14, and 19. There exists a number k such that $6k<n<6(k+ 1)$.By the Pigeonhole Principle, at least k+ 1 vertices of the n-gon have the same color. Between any two of these vertices are at least 4 others, because any 5 consecutive vertices have different colors. Hence, there are at least 5k+ 5 vertices, and $n \geq 5k+ 5$. However, this inequality fails for n= 7,8,9,13,14,19, a contradiction. Hence, a coloring is possible for all $n\geq 5$ except 7, 8, 9, 13, 14, and 19.

So final answer is 19

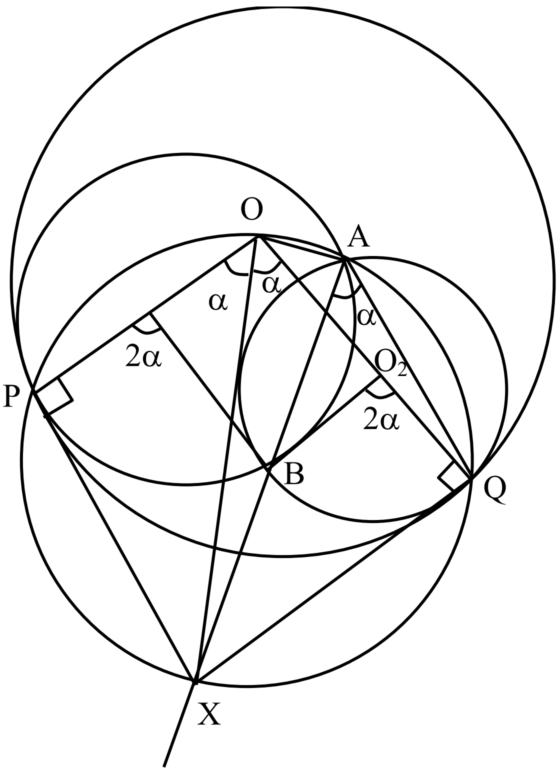
30. Let S be a circle of radius 10 with centre O. Suppose S_1 and S_2 are two circles which touch S internally and intersect each other at two distinct points A and B. If $\angle OAB = 90°$ what is the sum of the radii of S_1 and S_2?

Ans. (10)

Sol. Let S_1 & S_2 are tangent to S and P and Q respectively.

$\Rightarrow$ Tangents at P and Q to S and common chord of S_1S_2 are concurrent at X which is radical centre at S, S_1 and S_2.

Now XPOA & OPXQ are cyclic



$\Rightarrow$ P, O, A, Q and X are concylic

OP = OQ

XP = XQ

$\angle OPX = \angle OQX = 90°$

$\Rightarrow$ $\angle XOP + \angle XOQ = \alpha$ (let)

$\Rightarrow$ $\angle XAQ = \angle XOQ = \alpha$ (XPOAQ is cyclic)

$= \angle XAQ = \angle BAQ = \frac{1}{2} \angle BO_2Q$ (inscribed angle $= \frac{1}{2}$ central angle)

$\Rightarrow$ $\angle BO_2Q = 2\alpha$

Similarly $\angle BO_1P = 2\alpha$

$\Rightarrow$ $O_1B \| OO_2$ and $OO_1 \| O_2B$

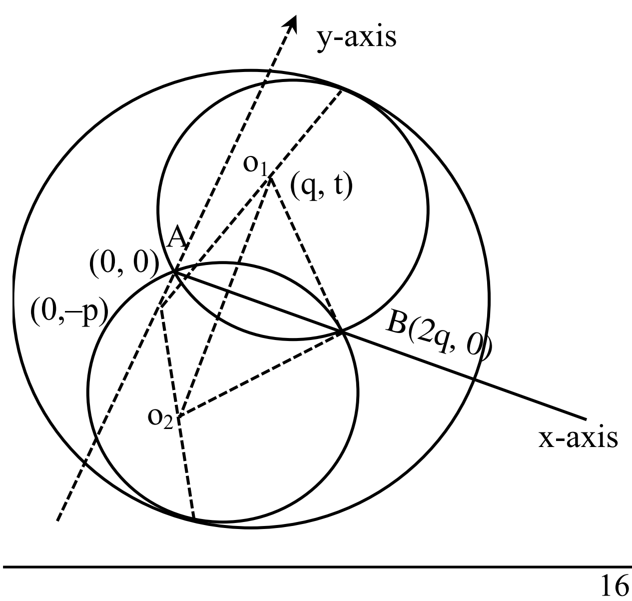
$\Rightarrow$ OO_1BO_2 is a parallelogram

$\Rightarrow$ $OO_1 = O_2B = r_2$ and $O_1P = r_1$

$\Rightarrow$ $OP = OO_1 + O_1P$

$\Rightarrow$ $10 = r_1 + r_2$

OR



$r_1 = \sqrt{q^2 + t^2} \quad \Rightarrow \quad r_1^2 = q^2 + t^2$

as distance from $(OO_1 + r_1) = 10$

$\sqrt{q^2 + (t+q)^2} + \sqrt{q^2 + t^2} = 10$

$\sqrt{q^2 + t^2 + p^2 + 2pt} + r = 10$

$\sqrt{p^2 + r_1^2 + +2p\sqrt{r^2 - q^2}} + r = 10$

$p^2 + r_1^2 + 2p\sqrt{r^2 - q^2} = r^2 + 100 - 20r$

$2p\sqrt{r^2 - q^2} = 100 - 20r - p^2$

$4p^2r^2 - 4p^2q^2 = 10000 + 400r^2 + p^4 - 4000r + 40r\,b^2 - 200p$

$r^2(4p^2 - 400) + r(4000 - 40p^2) - 4p^2q^2 - 10000 + 200p^2 - p^4 - 400r^2 = 0$

roots are r_1 and r_2

$r_1 + r_2 = \frac{4000 - 40p^2}{400 - 4p^2} = \frac{10(400 - p^2)}{400 - p^2} = 10$

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