40th INDIAN NATIONAL MATHEMATICAL OLYMPIAD-2026

Date: 18/01/2026

Max. Marks: 102 **Time allowed: 4.5 hours**

SOLUTIONS

1. Let $x_1, x_2, x_3, \ldots\ldots\ldots\ldots$ be a sequence of positive integers defined as follows : $x_1 = 1$ and for each $n \geq 1$ we have

$$x_{n+1} = x_n + \left[\sqrt{x_n}\right].$$

Determine all positive integers m for which $x_n = m^2$ for some $n \geq 1$. (Here [x] denotes the greatest integer less or equal to x for every real number x.)

Sol. Suppose for some n

$x_n = m^2$

Then $\left[\sqrt{x_n}\right] = m$ & hence $x_{n+1} = m^2 + m$

As long as $m^2 \leq x_k < (m+1)^2$

We have $\left[\sqrt{x_k}\right] = m$, and hence $x_{k+1} = x_k + m$

$\therefore$ While the sequence stays in the interval $[m^2, (n+1)^2)$, the term are of the form

$$X_k = m^2 + rm \qquad (r = 0, 1, 2, \ldots\ldots.)$$

It the sequence reaches another perfect square before leaving this interval, then for some integer k, then $m^2 + rm = k^2$

$k^2 - m^2 = rm$

$(k - m)\, k + m) = rm$

This equation has a positive integer solution only when $k = 2m$. Hence the next square reached after m^2 is $(2m)^2$

Since $x_1 = 1 = 1^2$

Repeating the above argument gives

$1^2 \rightarrow 2^2 \rightarrow 4^2 \rightarrow 8^2 \rightarrow 16^2 \rightarrow \ldots\ldots\ldots\ldots..$

$\therefore$ $x_n = m^2$ for some $\boxed{m = 2^k}$ $(k \geq 0)$

2. Let $f : N \rightarrow N$ be a function satisfying the following condition: for each $k > 2026$, the number f(k) equals the maximum number of times a number appears in the list $f(1), f(2), \ldots\ldots. f(k-1)$. Prove that $f(n) = f(n + f(n))$ for infinitely many $n \in N$.

(Here N denotes the set $\{1, 2, 3, \ldots\ldots.\}$ of positive integers.)

Sol. Let ℓ be the no. that appears m_l no. of times till

$\Rightarrow f(n) = \max(m_\ell) = m$ (for convenicnce)

$f(n)$ is non-decreasing function beyond $n = 2027$ as it is the made of the data so far, the only new numbers appearing in the data will be the numbers, who's frequency goes up, since the function has first 2026 values not necessarily following the rule beyond 2026, it can have any large maximum value. Let $N = \max\{f(k) \mid k \in \{1, 2, \ldots.2027\}|\}$

N + 1 itself appears 0 times in {f(1), f(2) …. f(2027)}

Since f(n) is non-decreasing we claim that the value does not become a constant beyond any finite n > 2027

FTSOC

let $\exists$ n > 2027 s.t. m = f(n) = f(n + 1) …. ∞

thus to calculate f(n + m) we must find at least m copies of m in the list before.

$\Rightarrow f(n + m + 1) > f(n) = m$

$\Rightarrow f$ will eventually be as large as N + 1 (contradiction)

Now let n be the first occurrence of N + 1

Since N + 1 appears 0 times in {f(1), f(2), …. f(n)}

$f(n) = f(n + 1) = f(n + 2) \ldots f(n + N) = N + 1$ Now N + 1 appears N + 1 time $\Rightarrow f(n + (N + 1)) = N + 1$

$\Rightarrow f(n + f(n)) = f(n)$

This will remain true for all n such that f(n) is the first occurrence of an integer in {f(1), f(2), …. f(n)}, n > 2027 and f(n) > N

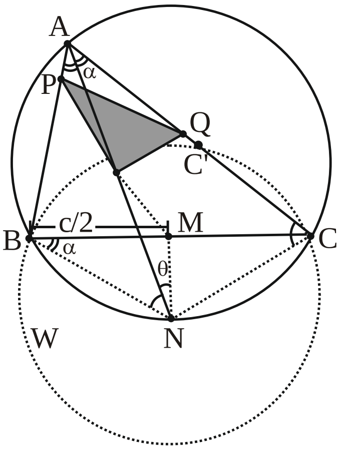
3. Let ABC be an acute angled scalene triangle with circumcircle Γ. Let M be the midpoint of BC and N be the midpoint of the minor arc BC of Γ. Points P and Q lie on segments AB and AC respectively such that BP = BN and CQ = CN. Point K ≠ N lies on line AN with MK = MN. Prove that ∠PKQ = 90.

Sol. W.L.O.G let AB < AC.

N $\rightarrow$ midpoint of incentre (I) and excenter (I_A) of ΔABC

So, N is circumcentre of ΔBIC.

Let circumcircle of ΔBIC be 'W'



$Pow_w A = AN^2 - r_w^2$

$= AN^2 - BN^2$

$= AC \cdot AC'$

$= AC \cdot AB = bc$

$Q = \angle ANM = \left|\frac{B-C}{2}\right|$

$NK = 2MN \cos\left|\frac{B-C}{2}\right|$

$= 2\,BN \sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)$

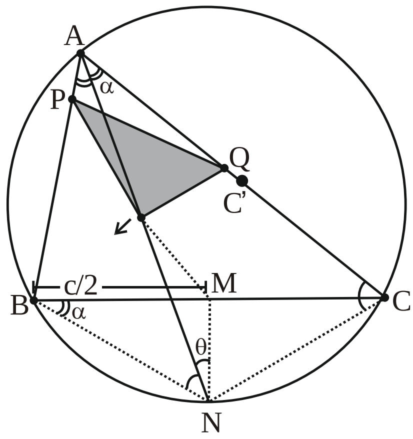
In $\underline{\Delta}$ABN by sine rule

$$\frac{AN}{\sin\left(\frac{A}{2}+B\right)} = \frac{BM}{\sin\left(\frac{A}{2}\right)} \Rightarrow \frac{AM}{BN} = \frac{\sin\left(90^\circ + \left(\frac{B-C}{2}\right)\right)}{\sin\left(\frac{A}{2}\right)}$$

$$AN \sin(A/2) = BN \cos\left(\frac{B-C}{2}\right)$$

$\therefore$ Nk = 2AN $\sin^2\left(\frac{A}{2}\right)$

So, AK = AN$\left(1-2\sin^2\left(\frac{A}{2}\right)\right)$ = AN·cosA



Let AP = p ⇒ AB – BP = c – BN = p

AQ = q ⇒ AC – CQ = b – CN = b – BM = q

C–BN b–CN = b–BM

$PQ^2 = AP^2 + AQ^2 - 2AP\cdot AQ\cos A$

$PK^2 = AP^2 + AK^2 - 2AP\cdot AK\cos\left(\frac{A}{2}\right)$

$QK^2 = AQ^2 + AK^2 - 2AQ\cdot AK\cos\left(\frac{A}{2}\right)$

For $PQ^2 = PK^2 + QK^2$

We show prove :

$2AK^2 - 2AK\cos\left(\frac{A}{2}\right)(p+q) = -2pq\cos A$

$2AN^2\cos^2 A - 2AN\cos A\cos(A/2)(p+q) = -2pq\cdot\cos A$

We show effectively prove that:

$AN^2\cos A - AN\cos\left(\frac{A}{2}\right)\cdot(p+q) + pq = 0$

By Substituting

$$\begin{cases} p+q = b+c-2BN \\ pq = bc - BN(b+c) + BN^2 \\ b+c = 2AN\cos A \end{cases}$$

Proves the required result.

4. Two integers a and b are called companions if every prime number p either divides both or none of a, b. Determine all functions $f:\mathbb{N}_0 \to \mathbb{N}_0$ such that $f(0) = 0$ and the numbers $f(m) + n$ and $f(n) + m$ are companions for all m, n $\in N_0$.

(Here N_0 denotes the set of all non-negative integers.)

Sol. We require a function f : $N_0 \to N_0$ s.t.

(i) rad(f(n) + m) = rad(f(m) + n)

(ii) f(0) = 0

(iii) rad(n) = largest square free divisor of n

i.e. if $n = p_1^{\alpha_1}, p_2^{\alpha_2}, ...p_k^{\alpha_k}$ $p_i \in$ Primes, $\alpha_i \in Z^+$

⇒ rad(n) = $p_1p_2 \ldots p_k$ here all pi's are distinct

Claim 1 : $q \nmid f(p)$ where p and q are distinct primes.

proof n = p and m = 0 gives

rad (f(n)) = rad(n) ⇒ rad(f(p)) = rad(p) = p

thus since $q \nmid p \Rightarrow q \nmid f(p)$

Claim 2 : $f(p) = p\ \forall$ p primes not of the form $2^m - 1$.

proof : since $q \nmid f(p)\ \forall$ distinct prime p, q

$f(p) = p^k$ where $k \in Z^+$

let n = p m = 1 $\Rightarrow rad(f(p + 1)) = rad\ (p + 1)$

$rad(p^k + 1) = rad\ (p + 1)$

by Zsigmondy's theorem for $k > 2\ \exists$ at least 1 prime divisor of $p^k + 1$ that doesn't divide p + 1 contradiction.

$\Rightarrow f(p) = p\ \forall$ p that are not 1 less than a power of 2

Claim 3 : $f(p) = p\ \ \forall$ p of the form $2^m - 1$

by Dirichlet's them

infinitely many primers r exist s.t. $-p = r \bmod q$ for p, q primes

$\Rightarrow$ let m = r, n = p

If $f(p) = 1 \Rightarrow q|r + p$ and $q|r + 1$

$\Rightarrow q|p - 1$ but q was an arbitrary prime, contradiction

$\Rightarrow f(p) = p\ \forall$ primes p.

Claim 4 : $f(n) = n\ \forall\ n \in Z^+$

Proof :

Consider an arbitrary large prime q such that

$(-n) \bmod q = p$ (Dirichlet's theorem)

Where p is a prime.

For m = p

$rad\ (f(n) + p) = rad(n + p)$

$q \mid f(n) - n$

since of every sufficiently large q

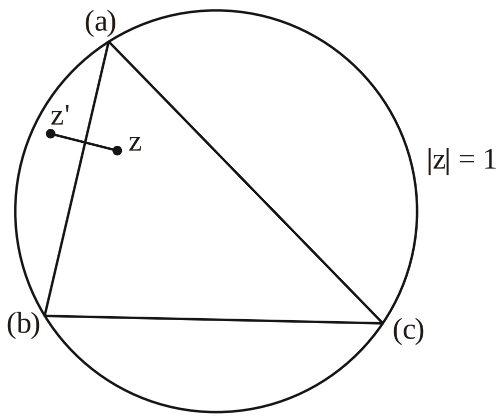
$f(n) = n \bmod q$

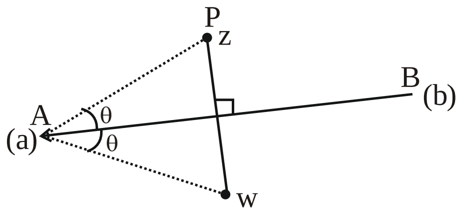
f(n) = n identically.

5. Three lines ℓ_1, ℓ_2, ℓ_3 form an acute angled triangle T in the plane. Point P lies in the interior of T. Let T_i denote the transformation of the plane such that the image T_i (X) of any point X in the plane is the reflection of X in ℓ, for each $i \in \{1, 2, 3\}$. Denote by P_{ijk} the point $T_k(T_j(T_i(P)))$ for each permutation (i, j, k) of (1, 2, 3).
Prove that $P_{123}, P_{132}, P_{213}, P_{231}, P_{312}, P_{321}$ are concyclic if and only if P coincides with the orthocentre of T.

Sol. orthocentre $\rightarrow a + b + c$

circumcenter $\rightarrow O + Oi$





Reflection of P(z) in segment joining a & b or |z| = 1.

$$\frac{z-a}{b-a}=\frac{AP}{AB}\ e^{i\theta}$$

$$\frac{w-a}{b-a}=\frac{AP}{AB}\ e^{-i\theta}$$

$$\frac{w-a}{b-a}=\overline{\left(\frac{z-a}{b-a}\right)}$$

$$\frac{w-a}{b-a}=\frac{\bar{z}-\bar{a}}{\bar{b}-\bar{a}} \qquad \bar{a}=\frac{1}{a},\bar{b}=\frac{1}{b}$$

$$\frac{w-a}{b-a}=\frac{ab\bar{z}-b}{a-b} \qquad \Rightarrow w-a=b-ab\bar{z}$$

$$\boxed{w=a+b-ab\bar{z}}$$

$P_{123}(z) = T_3(T_2(T_1(z)))$ $\qquad$ $H = a + b + c$

$$T_1(H) = b + c - bc\left(\bar{a}+\bar{b}+\bar{c}\right) = b + c - bc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=-\frac{bc}{a}$$

$$T_2(T_1(H)) = c + a + ca\left(\frac{\bar{b}\bar{c}}{\bar{a}}\right) = c + a + ca\cdot\frac{c}{bc} = a + c + \frac{a^2}{b}$$

$$P_{123} = T_3(T_2(T_1(H))) = a+b-ab\left(\frac{1}{a}+\frac{1}{c}+\frac{b}{a^2}\right)$$

$$=a+b-b-\frac{abc}{c}-\frac{b^2}{a}=a-\frac{ab}{c}-\frac{b^2}{a}$$

$$d_{123}=d\left(P_{123},H\right)=\left|a+b+c-a+\frac{ab}{c}+\frac{b^2}{a}\right|$$

$$d_{123}=\left|b+c+\frac{ab}{c}+\frac{b^2}{a}\right|$$

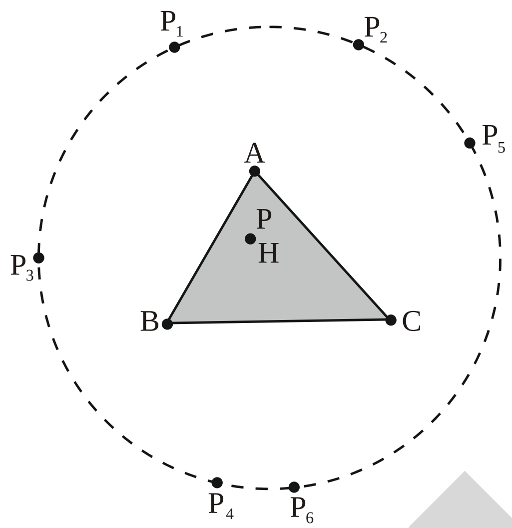
$$=\left|a^2b+b^2c+c^2a+a^3c\right|$$

Similarly, $d_{321}=\left|c^2b+b^2a+a^2c+abc\right|$

$$=\left|\overline{c^2b+b^2a+a^2c+abc}\right|$$

$$=\left|\frac{1}{c^2b}+\frac{1}{b^2a}+\frac{1}{a^2c}+\frac{1}{abc}\right|$$

$$= \frac{\left|a^2b + c^2a + b^2c + abc\right|}{\left|a^2b^2c^2\right|} = \frac{d_{123}}{1}$$



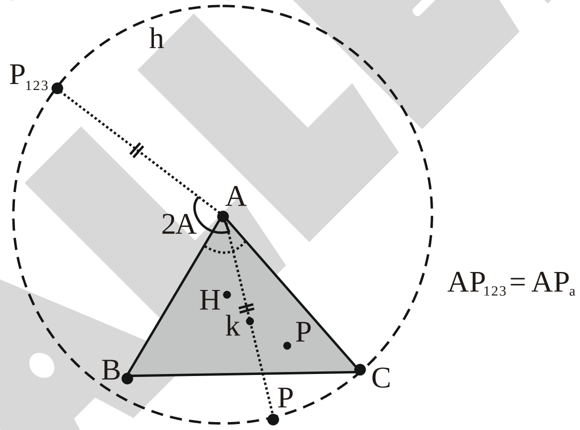
Similar logic can be applied to remaining permutations.

for P = H, the permutation of points obtained are concyclic.

Converse if $P \neq H$

Let P_a be reflection of P in side BC. The point P_{123} is obtained by reflecting P_a in AC & then in AB.

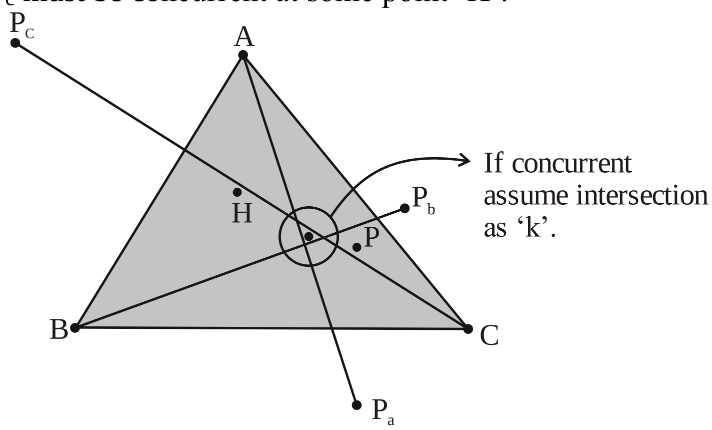
P_{123} is effectively image of P_a rotated by +2A around A.



So, P_{123} & P_{132} are symmetric rotation of P_a about A in opposite direction. So, $\perp^r$ bisector of $P_{123}P_{132}$ is AP_a.

For points to be concyclic center must be on 1st bisector.

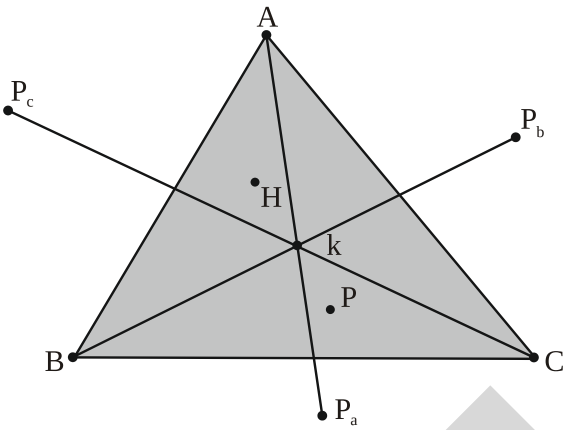
$\therefore$ AP_a, BP_b, CD_c must be concurrent at some point 'K'.

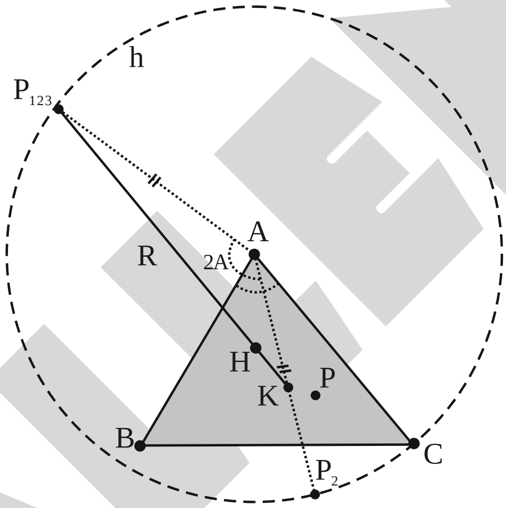


$AP_{132} = AP_{123} = AP_a = \ell_a$

$BP_{213} = BP_{231} = BP_b = \ell_b$

$CP_{312} = CP_{321} = CP_c = \ell_c$





R = radius of circle through $P_{i,j,k}$'s permutations.

$P_a = b + c - bc\overline{p}$

Solving like before

$$P_{123} = a + b - \frac{ab}{c} + \frac{b^2}{c} - b^2\overline{p}$$

$$\& \ P_{213} = a + b - \frac{ab}{c} + \frac{a^2}{c} - a^2\overline{p}$$

$$P_{123} - P_{213} = \left(\frac{b^2}{c} - b^2\overline{p}\right) - \left(\frac{a^2}{c} - a^2\overline{p}\right) = \left(b^2 - a^2\right)\left(\frac{1}{c} - \overline{p}\right) \neq 0 \qquad (P \neq C \ \& \ a \neq b)$$

$\therefore P_{123} - H \ \& \ P_{213} - H$

must relate by rotation or reflection

Solving : $\left|P_{123} - H\right|^2 = \left|P_{213} - H\right|^2 \Rightarrow p - ab\overline{p} = c - \dfrac{ab}{c}$ … (1)

Similarly, $|P_{132} - H|^2 = |P_{312} - H|^2 \Rightarrow p - ac\overline{p} = b - \dfrac{ac}{b}$ … (2)

From (1) & (2)

$$-ab\bar{p}+ac\bar{p}=c-b-\frac{ab}{c}+\frac{ac}{b}$$

$$\Rightarrow \quad \bar{p}a(c-b)=(c-b)+\frac{a}{bc}(c-b)(c+b) \qquad (c\neq b)$$

$$\Rightarrow \quad p\bar{a}=1+\frac{a(b+c)}{bc}=1+a\left(\frac{1}{b}+\frac{1}{c}\right)$$

$$\Rightarrow \quad p\bar{a}=1+\bar{a}(b+c)$$

$$p=\frac{1}{\bar{a}}+b+c=a+b+c$$

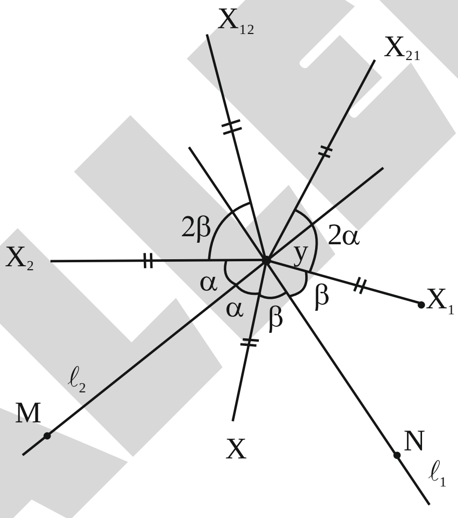
So, P can be related to H only if P ≡ H.

Alternate Solution :

Let P_{ijk} Be the point $T_k(T_j(T_i(p)))$

Claim : for any point X and lines ℓ_1, ℓ_2

X_{12} and X_{21} are symmetric about the line $\overline{X(\ell_1 \cap \ell_2)}$



Let $Y = \ell_1 \cap \ell_2$

$YX_{21} = YX_2 = YX = YX_1 = YX_{12}$

if $< XYM = \alpha$, $\angle XYN = \beta$

$\Rightarrow$ $\angle XYX_{21} = 2(\alpha + \beta) = \angle XYX_{12}$

$\Rightarrow$ X_{12} and X_{21} are symmetric about XY

$\Rightarrow$ XY is $\perp$ bisector of $X_{12}X_{21}$

$\Rightarrow$ let $\ell_1 \cap \ell_2 = C$ $\qquad T_1(A) = A'$

$\ell_2 \cap \ell_3 = A$ and $T_2(B) = B'$

$\ell_1 \cap \ell_3 = B$ $\qquad T_3(C) = C'$

$\perp$ bisectors of the following 9 points of points is mentioned below.

$P_{123}P_{132} \rightarrow AP_1$	$P_{23}P_{32} \rightarrow AP \Rightarrow P_{231}\,P_{321} \rightarrow A'P_1$
$P_{213}P_{231} \rightarrow BP_2$	$P_{13}P_{31} \rightarrow BP \Rightarrow P_{132}\,P_{312} \rightarrow B'P_2$
$P_{312}P_{321} \rightarrow CP_3$	$P_{12}P_{21} \rightarrow CP \Rightarrow P_{123}\,P_{213} \rightarrow C'P_3$

Since $A'P_1 \cap AP_1 = \begin{cases} P_1 \text{ if } P \text{ does not lie on } \perp \text{ from A} \\ \overline{AP} \text{ if } P \text{ lie on } \perp \text{ from A to BC} \end{cases}$

Similarly for $B'P_2 \cap BP_2$ and $C'P_3$ and CP_3

If P does not coincide with the orthocentre of ΔABC then the 6 $\perp$ bisectors AP_1, $A'P_1$, BP_2, $B'P_2$, CP_3, $C'P_3$ all meet at 3 distinct points outside the ΔABC

Hence the 6 points cannot be concyclic.

If P lies on the orthocentre then the following 3 sets of 3 lines are identical

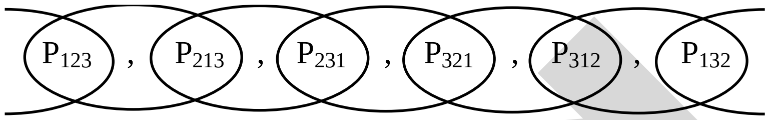
$AP = A'P_1 = AP_1$

$BP = B'P_2 = BP_2$

$CP = C'P_3 = CP_3$

$\Rightarrow$ the common points of intersection is P.

Consider the 6 points P_{ijk} where i, j, k are permutation of 1, 2, 3



Circumcentre of any 3 consecutive points in this list is the point P thus P is equidistant from all 6 point and all these 6 points are concyclic.

6. Two decks A and B of 40 cards each are placed on a table at noon. Every minute thereafter, we pick the top cards $a \in A$ and $b \in B$ and perform a duel.

For any two cards $a \in A$ and $b \in B$, each time a and b duel, the outcome remains the same and is independent of all other duels. A duel has three possible outcomes :

If a card wins, it is placed back at the top of its deck and the losing card is placed at the bottom of its deck.

If a and b are evenly matched, they are both removed from their respective decks.

If a and b do not interact with each other, then both are placed at the bottom of their respective decks.

The process ends when both decks are empty. A process is called a game if it ends. Prove that the maximum time a game can last equals 356 hours.

Sol. Assume a game of lattice hopping being played on a grid for lattice points x, y where $1 \le x, y \le N$ the process of card comparision in the question can be mapped to lattice jumps.

(i) If deck A cord wind then the equivalent move becomes.

$(x, y) \to (x, y + 1)$ number x, y are mod N

(ii) If deck B cord wins then

$(x, y) \to (x + 1, y)$

(iii) If no interaction then $(x, y) \to (x + 1, y + 1)$

(iv) If equal then the (x_0, y_0) must be highlighted on reaching such a point both cards are removed from the deck this equals to removed of all points (x_0, y), $y \in \{1, 2, \ldots. N\}$ and (x, y_0), $x \in \{1, 2, \ldots. N\}$,

2N – 1 points and $N \to N - 1$.

Now all points are re-numbered $(x, y) \to (f(x, x_0), f(y, y_0))$

$$\text{where } f(n, n_0) = \begin{cases} n; & n < n_0 \\ \text{D.N.E.}; & n = n_0 \\ n-1; & n > n_0 \end{cases}$$

In order to find an upper bound, we need to have at least n non-attacking rooks position on an n × n board such that these are the highlighted positions thus this leaves $n^2 - n + 1$ maximal moves before making a move into a highlighted lattice point and effectively resetting the lattice board.

$$\Rightarrow \text{Max moves of a terminating process } \le \sum_{n=1}^{N} n^2 - n + 1$$

$$= \frac{N(N+1)(2N+1)}{6} - \frac{N(N+1)}{2} + N$$

$$= \frac{N(2N^2 + 3N + 1 - 3N - 3 + 6)}{6}$$

$$= \frac{N(N^2+2)}{3}$$

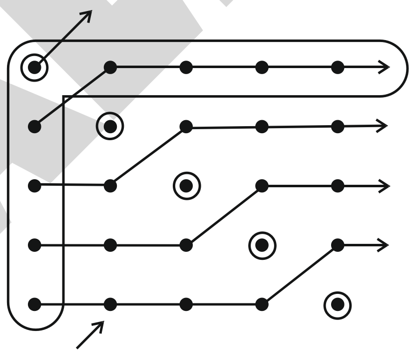
for N = 40 this upper bound is $\frac{40 \times 1602}{3}$ min

$$= \frac{40 \times 1602}{3 \times 60} \text{ hr}$$

$$= 2 \times \frac{1602}{9} = \boxed{356}$$

Construction : Let the highlighted points be (x, y) where x + y = N + 1

let the path avoiding all these highlighted points be as in the diagram below, for N = 5



$$M(x,y) = \begin{cases} (x+1, y); & x+y \ne N \text{ or } N+1,\ x \ne N \\ 1, y; & x = N \\ (x+1, y+1); & x+y = N \\ (x,y); & x+y = N+1 \text{ highlighted pts} \\ \text{with board reset} & \end{cases}$$

The path wind around all highlighted pts and covers $N^2 - N$ pts, starting at (1, 1) and ending at (1, N) after which the entire first column and top row are delated and we return to 2, 1 which is 1, 1 by default.

Thus total moves = $\sum_{n=40}^{1} n^2 - n + 1$ = 21360 min = 356 hrs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Final Round Result:

57th National Mathematics Talent Contest (NMTC) 2025-26

Conducted by The Association of Mathematics Teachers of India (AMTI), Chennai

36

ALLEN students secured
Top 20 Ranks (Primary, Sub-Junior, Junior & Inter) in NMTC Final

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Heartiest Congratulations

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Result: National Standard Examination (NSE) 2025-26

(Stage 1 of Official International Science Olympiad 2026)

637*

Students Selected for

Indian National Olympiads (INO 2026 - Stage 2)

*As per result compiled so far

138 students for **INPhO**	**189** students for **INChO**	**143** students for **INAO**	**73** students for **INBO**	**94** students for **INJSO**

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